UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 2
Revision of vector calculus

- Gradient, divergence, curl and Laplace operators
- The del symbol
- Useful vector calculus theorems
- Laws of electrostatics and magnetostatics in the differential form

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Previous lecture

❖ Laws of electrostatics and magnetostatics *in free space*, *in the integral form*, useful for practical computations:

Gauss law (*universally valid*)

$$\int_S \vec{E} d\vec{S} = \frac{1}{\varepsilon_0} \int_V \rho dV$$

Conservative nature of the **E**-field (*static field only*)

$$\oint_L \vec{E} d\vec{l} = 0$$

Absence of magnetic poles (*universally valid*)

$$\int_S \vec{B} d\vec{S} = 0$$

Ampere's law (*static field only*)

$$\oint_L \vec{B} d\vec{l} = \mu_0 \int_S \vec{j} d\vec{S}$$

Gradient of a scalar field

Gradient of a **scalar field** (**grad**φ) is a **vector field**:

- ✓ direction is perpendicular to equipotential surfaces;
- ✓ magnitude is equal to the derivative along that direction.

For any unit vector $\vec{s}$, $\dfrac{\partial \varphi}{\partial s} = (\vec{s}\,\mathrm{grad}\varphi)$

Equivalently, difference in field values at two nearby points:

$$\Delta\varphi = (\mathrm{grad}\varphi)\delta\vec{r}$$

(scalar product of the gradient and the vector displacement).

Therefore the *slope is steepest* along the direction of gradient.

We know from calculus: $\Delta\varphi = \dfrac{\partial\varphi}{\partial x}\Delta x + \dfrac{\partial\varphi}{\partial y}\Delta y + \dfrac{\partial\varphi}{\partial z}\Delta z$

Therefore in *any* Cartesian coordinates,

$$\mathrm{grad}\varphi = \vec{e}_x\frac{\partial\varphi}{\partial x} + \vec{e}_x\frac{\partial\varphi}{\partial y} + \vec{e}_x\frac{\partial\varphi}{\partial z}$$

Divergence & curl of a vector field

Divergence of a **vector field** is a **scalar field**:

$$\mathrm{div}\vec{F} = \lim_{V\to 0}\left(\frac{1}{V}\int_{S(V)}\vec{F}d\vec{S}\right)$$

(volume density of flux)

Curl of a **vector field** is a **pseudovector field**

(changes direction in a mirror image, defined in three dimensions only):

$$\mathrm{curl}\vec{F} = \lim_{V\to 0}\left(\frac{1}{V}\int_{S(V)}(d\vec{S}\times\vec{F})\right)$$

(volume density of overall circulation)

Equivalently, projection of the curl onto any unit vector $\vec{s}$:

$$(\mathrm{curl}\vec{F})\cdot\vec{s} = \lim_{A\to 0}\left(\frac{1}{A}\oint_{L}\vec{F}d\vec{l}\right)$$

where the loop **L** lies in the plane orthogonal to $\vec{s}$

The del symbol

The **del** symbol (also known as **nabla**):

in *Cartesian coordinates*, $\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$

Gradient of a **scalar field** is a **vector field:**

$$\mathrm{grad}\ \varphi = \nabla\varphi = \left(\frac{\partial \varphi}{\partial x}\vec{e}_x + \frac{\partial \varphi}{\partial y}\vec{e}_y + \frac{\partial \varphi}{\partial z}\vec{e}_z\right)$$

Divergence of a **vector field** is a **scalar field:**

$$\mathrm{div}\,\vec{F} = \nabla\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Curl of a **vector field** is a **pseudovector field**:

$$\mathrm{curl}\ \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} =$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\vec{e}_z$$

The Laplace operator

Second derivatives of the fields: the *Laplace operator*

$$\nabla^2 = \left(\vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}\right)^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

For a **scalar field**, the result is a **scalar field**:

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

For a **vector field**, the result is a **vector field**:

$$\nabla^2 \vec{F} = \nabla^2 (\vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z) =$$

$$= \vec{e}_x \nabla^2 F_x + \vec{e}_y \nabla^2 F_y + \vec{e}_z \nabla^2 F_z$$

Useful theorems (prove them!)

(T1) "div curl = 0": $\nabla(\nabla \times \vec{F}) = 0$

(T2) "curl grad = 0": $\nabla \times (\nabla\varphi) = \vec{0}$

(T3) "curl curl = grad div – del squared"

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla\vec{F}) - \nabla^2\vec{F}$$

Computation of divergence:

It is often convenient to use Cartesian coordinates:

$$\nabla\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Unit radial vector

For a radially/spherically symmetric vector field, $\vec{F} = F_r(r) \cdot \vec{\hat{r}}$,

(T4) in cylindrical coordinates (r,θ,z), $\nabla\vec{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r}$

(T5) in spherical coordinates (r,θ,φ), $\nabla\vec{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r}$

Vector calculus theorems

Relating physics inside a volume to physics at its surface, or physics on a surface to physics on its boundary:

Divergence (Gauss) theorem:

$$\int_S \vec{F} d\vec{S} = \int_V \nabla \vec{F} dV$$

Flux of a vector field through a closed surface

Integral of its divergence over the enclosed volume

Curl (Kelvin–Stokes) theorem:

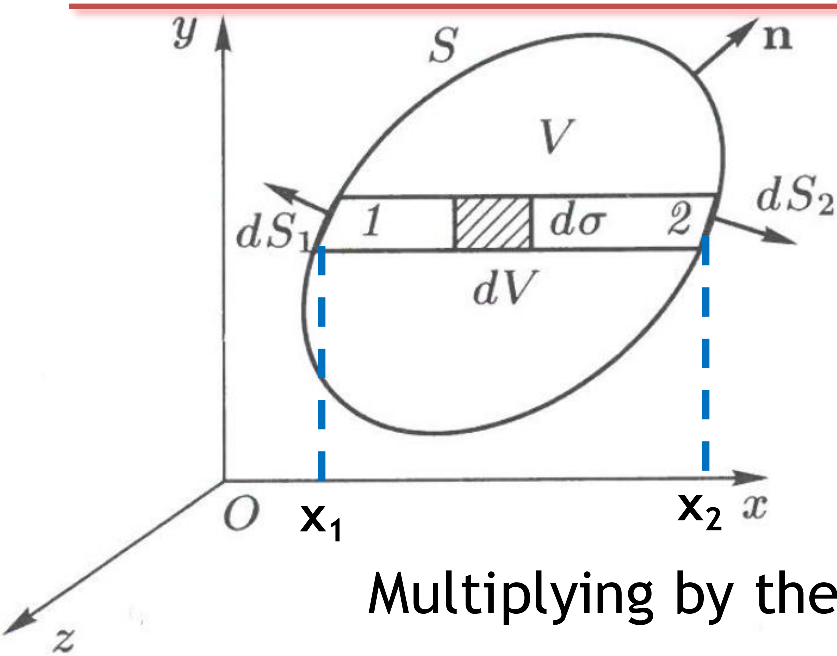
$$\oint_L \vec{F} d\vec{l} = \int_S (\nabla \times \vec{F}) d\vec{S}$$

Line integral of a vector field around a closed curve

Flux of its curl through **any surface** enclosed by the curve

Proof of divergence theorem (1)

[not discussed in the lecture]



For a scalar field $f(x, y, z)$,

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial x} dx = f_2 - f_1$$

Multiplying by the cross-section $d\sigma > 0$,

and using $d\sigma = -n_{1x} dS_1 = n_{2x} dS_2$,

where $\vec{n}$ is a unit normal vector to surface,

$$\int_{dV} \frac{\partial f}{\partial x} dV = (f_2 - f_1) d\sigma = f_1 n_{1x} dS_1 + f_2 n_{2x} dS_2 = \int_{dS_1 + dS_2} f n_x dS$$

Summing over elementary volumes, $\displaystyle\int_V \frac{\partial f}{\partial x} dV = \int_S f n_x dS$

Proof of divergence theorem (2)

[not discussed in the lecture]

For a vector field $\vec{F}(x,y,z)$,

$$\int\limits_V \left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)dV=\int\limits_S (F_x n_x+F_y n_y+F_z n_z)dS$$

$$\int\limits_V \nabla\vec{F}dV=\int\limits_S (\vec{F}\vec{n})dS$$

Using the vector area $d\vec{S}=\vec{n}dS$,

$$\underbrace{\int\limits_S \vec{F}d\vec{S}}_{\text{Flux of a vector field through a closed surface}}=\underbrace{\int\limits_V \nabla\vec{F}dV}_{\text{Integral of its divergence over the enclosed volume}}$$

Flux of a vector field through a closed surface

Integral of its divergence over the enclosed volume

Differential laws for static fields

Gauss law and the divergence theorem for *any* volume V:

$$\int_S \vec{E} d\vec{S} = \frac{1}{\varepsilon_0} \int_V \rho dV = \int_V \nabla \vec{E} dV$$

Therefore $\nabla \vec{E} = \mathrm{div} \vec{E} = \rho / \varepsilon_0$; similarly $\nabla \vec{B} = \mathrm{div} \vec{B} = 0$

Conservative nature of the electrostatic field E, and curl theorem for *any* loop L and *any* surface enclosed S:

$$\oint_L \vec{E} d\vec{l} = \int_S (\nabla \times \vec{E}) d\vec{S} = 0$$

Therefore, for the electrostatic field, $\nabla \times \vec{E} = \mathrm{curl}\ \vec{E} = 0$

Similarly, Ampere's law becomes $\nabla \times \vec{B} = \mathrm{curl}\ \vec{B} = \mu_0 \vec{j}$

Field potentials

For the electrostatic field, $\nabla \times \vec{E} = 0$

For any scalar field **(T2)**, $\nabla \times (\nabla \varphi) = 0$

Therefore, one can define a *scalar potential* of the electrostatic field (the minus sign is a convention):

$$\vec{E} = -\nabla \varphi$$

The electrostatic field is *irrotational* or *conservative*.

For the magnetic field, $\nabla \vec{B} = 0$ (the field is *solenoidal*)

For any vector field **(T1)**, $\nabla (\nabla \times \vec{A}) = 0$

Therefore, one can define a *vector potential* of the magnetic field:

$$\vec{B} = \nabla \times \vec{A}$$

Definitions of the potentials are not unique ("gauge freedom").
More details: *lecture 3*.

Summary

- Laws of electrostatics and magnetostatics *in free space*, *in the differential form*:

Gauss law (*universally valid*)	Conservative nature of the E-field (*static field only*)
$\nabla\vec{E} = \rho/\varepsilon_0$	$\nabla \times \vec{E} = 0$
Absence of magnetic poles (*universally valid*)	Ampere's law (*static field only*)
$\nabla\vec{B} = 0$	$\nabla \times \vec{B} = \mu_0\vec{j}$

- Electrostatic field is described by a scalar potential, $\vec{E} = -\nabla\varphi$
- Magnetic field is described by a vector potential, $\vec{B} = \nabla \times \vec{A}$