UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 3
Ideal conductors; electrostatic problems

- Ideal conductors in the electrostatic field
- The Laplace equation of electrostatics
- The uniqueness theorem
- The method of image charges

Prof. Evgueni Goudzovski • Physics West 212 • goudzovs@cern.ch

Previous lecture

❖ Laws of electrostatics and magnetostatics *in free space*, *in the differential form*:

Gauss law (*universally valid*)

$$\nabla \vec{E} = \rho/\varepsilon_0$$

Conservative nature of the E-field (*static field only*)

$$\nabla \times \vec{E} = 0$$

Absence of magnetic poles (*universally valid*)

$$\nabla \vec{B} = 0$$

Ampere's law (*static field only*)

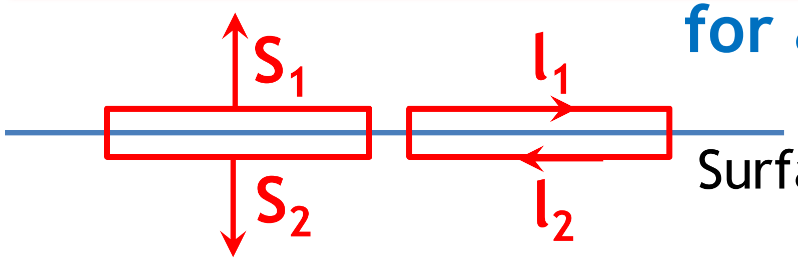
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

❖ Electrostatic field is described by a scalar potential, $\vec{E} = -\nabla\varphi$

❖ Magnetic field is described by a vector potential, $\vec{B} = \nabla \times \vec{A}$

Boundary conditions

for a thin electrically charged sheet



Surface charge density: σ [unit: C/m]

Gauss law for a very thin ("pillbox") cylinder S:

$$\int_S \vec{E} d\vec{S} = \frac{q}{\varepsilon_0}\text{, therefore } \vec{E}_1 \vec{S}_1 + \vec{E}_2 \vec{S}_2 = \frac{\sigma S}{\varepsilon_0}$$

Using $\vec{S}_1 = -\vec{S}_2$, we obtain for the normal field component:

$$E_{1n} - E_{2n} = \frac{\sigma}{\varepsilon_0}$$

For a very thin rectangular loop, $\oint_L \vec{E} d\vec{l} = \vec{E}_1 \vec{l}_1 + \vec{E}_2 \vec{l}_2 = 0$

similarly leads to $E_{1t} = E_{2t}$ for the tangential field component

Conductors in electrostatic field

Conductor: often a lattice of ions and a gas of free electrons. Due to the re-distribution of *free charges*, total electrostatic field inside *ideal conductor* in the *static case*: $\vec{E} = 0$.

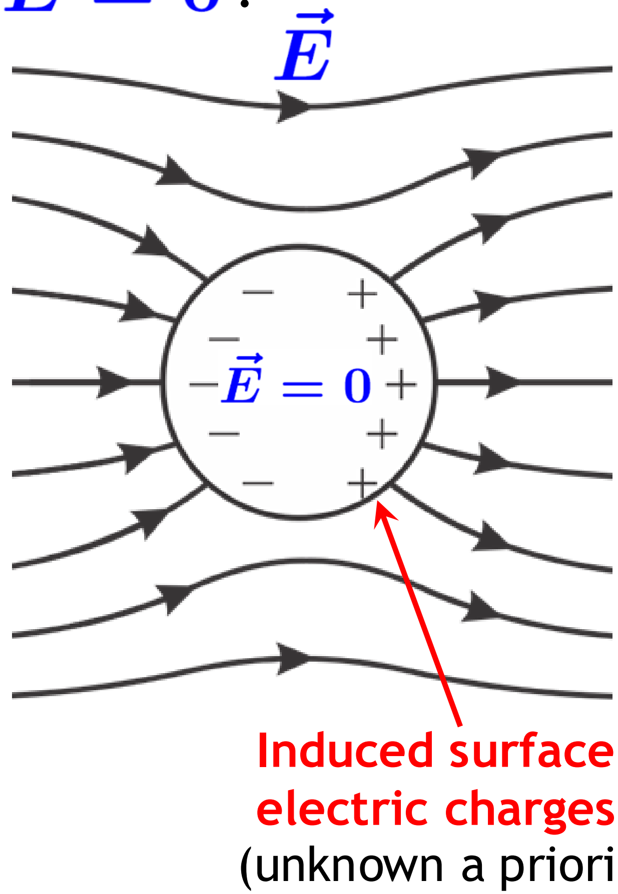
Conductor volume is equipotential: φ=const

1) Gauss law, any point inside a conductor:

$$\rho = \varepsilon_0 \nabla \vec{E} = \varepsilon_0 \nabla (\vec{0}) = 0$$

2) Conductor in an external static **E** field. It follows from the boundary conditions that the **field is perpendicular to the surface**:

$$E_n = \frac{\sigma}{\varepsilon_0} \quad \text{and} \quad E_t = 0$$



There are no volume charges inside a conductor (ρ=0). All charges are localised at the surface (σ≠0 in general).

The Laplace equation

Equations of electrostatics: $\nabla \vec{E} = \rho/\varepsilon_0\,;\ \nabla \times \vec{E} = 0$

It is possible to define the electrostatic potential, $\vec{E} = -\nabla\varphi$

considering that $\nabla \times (\nabla\varphi) = 0$ *(lecture 2)*

Electrostatic potential at a point **A**:
(integral along *any* line)

$$\varphi_A = \int_A^\infty \vec{E}d\vec{l}$$

Poisson equation in free space: $\nabla(\nabla\varphi) = \nabla^2\varphi = -\rho/\varepsilon_0$

Widely used in physics: description of heat flow, diffusion, …

In the absence of free charges, *Laplace equation*: $\nabla^2\varphi = 0$
(and polarisation charges, *lectures 7–8*)

Explicitly in the Cartesian coordinates,

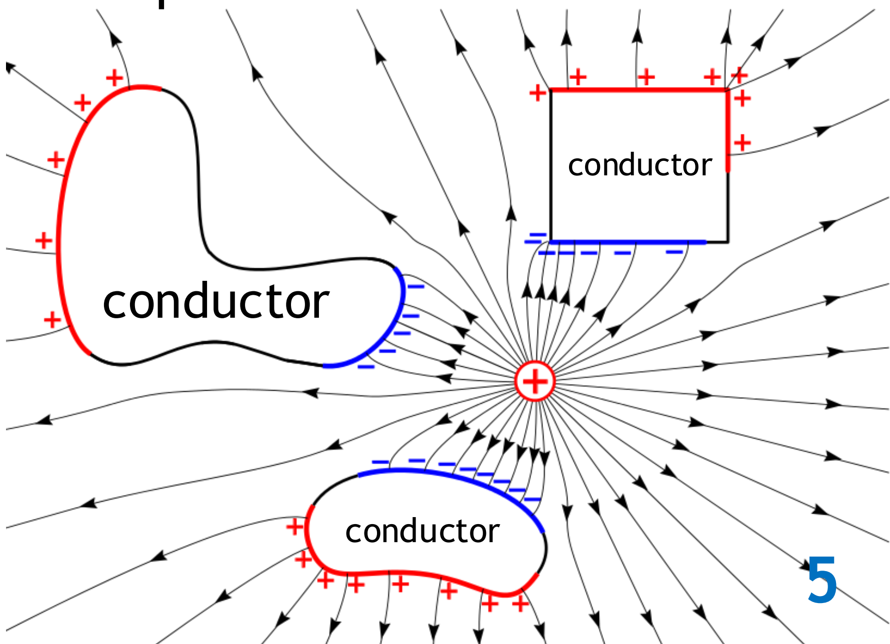
$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

Electrostatic computations

- Field computation for a *known distribution of charges*: Coulomb law and the principle of superposition *(lecture 1)*.
- But distribution of *free charges* in conductors is *not known* a priori.

The **general problem of electrostatics** (in free space):

- Several conducting bodies are placed in vacuum.
- *Boundary conditions*: for each conductor, either its potential φ_i or its total charge Q_i is specified.
- It is required to compute the electrostatic potential $\varphi(x,y,z)$ in all points of space.
- When $\varphi(x,y,z)$ is found, the electric field is computed as $\vec{E} = -\nabla\varphi$
- Densities of the induced charge at the conductor surfaces are found as $\sigma(x,y,z) = \varepsilon_0 E_n(x,y,z)$, where $E_n(x,y,z)$ is the field just outside the conductor.



The uniqueness theorem

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

A solution of Laplace equation does not have local maxima/minima (a local minimum, for example, means $\frac{\partial^2 \varphi}{\partial x^2} > 0, \ \frac{\partial^2 \varphi}{\partial y^2} > 0, \ \frac{\partial^2 \varphi}{\partial z^2} > 0$).

Consider two different solutions for the same set of boundary conditions:

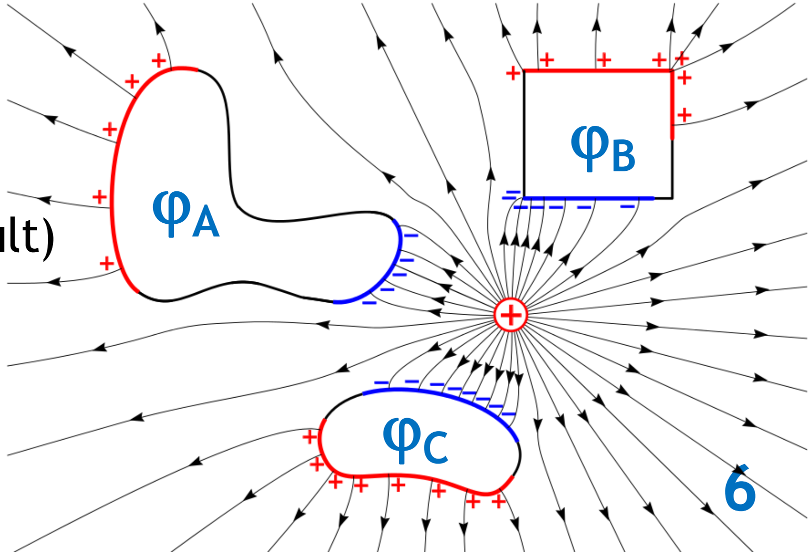
$\nabla^2 \varphi_1 = 0, \ \ \nabla^2 \varphi_2 = 0$; therefore $\nabla^2(\varphi_0) = \nabla^2(\varphi_1 - \varphi_2) = 0$

The solution $\varphi_0(\vec{r})$ has no minima/maxima;

the boundary conditions are $\varphi_0 = 0$

Therefore $\varphi_0(\vec{r}) \equiv 0$

(Proof for boundary conditions for Q_i is more difficult)



Poisson and Laplace equations have a unique solution for any complete set of boundary conditions.

Faraday cage

Independently of the method chosen, if a solution to $\nabla^2 \varphi = 0$ is found (or guessed), it is *the only solution*.

For a cavity in a conducting body: inside the cavity, $\nabla^2 \varphi = 0$

On the surface of the cavity, $\varphi = \varphi_0 = \mathrm{const}$

A (unique!) solution inside the cavity: $\varphi(\vec{r}) \equiv \varphi_0$

Therefore, no electric field in the cavity:

$$\vec{E} = \nabla \varphi_0 = \vec{0}$$

❖ "Faraday cages" are used for electrostatic shielding of equipment, in USB/coaxial cables, in forensics.

More generally, fields inside and outside a conductive enclosure are independent.

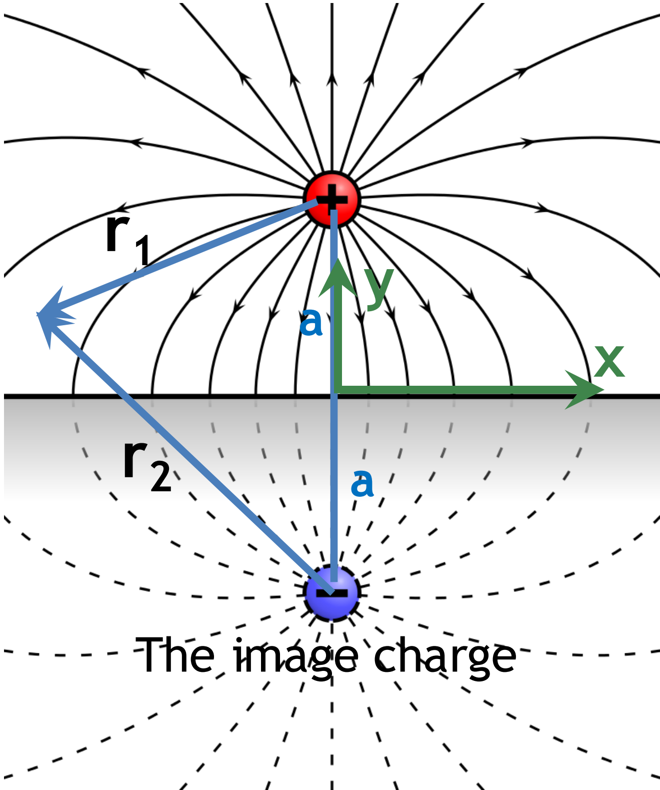
The method of image charges

For a system of charges and a grounded ($\varphi=0$) conductor:

- ❖ there are *induced charges* on the conductor surface;
- ❖ using the uniqueness theorem, the conductor (and induced charges) can be replaced by a system of *image charges* in such a way that the conductor surface remains at $\varphi=0$.

Example: a point charge **Q** at a distance **a** from an infinite conducting plate of any thickness.

Consider an *image charge* **–Q** placed symmetrically at a distance **a** below the surface.



Potential due to the real and image charges:

$$\varphi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

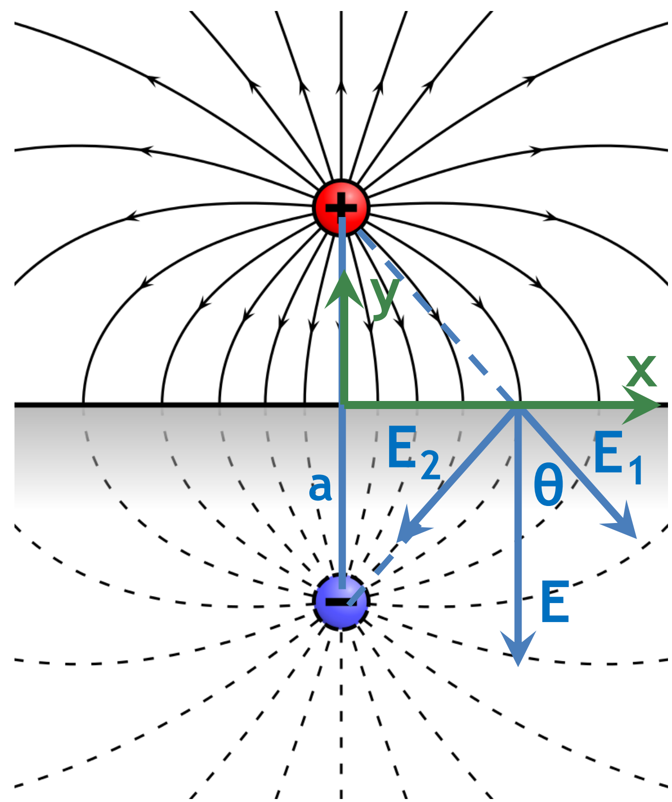
At the surface, $\varphi|_{y=0} = 0$:
the image charge is equivalent to the plate.

Example (continued)

Above the surface, the field of the *induced charges* is equivalent to the field of the *image charge*.

Attractive force between point charge and the plate:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{(2a)^2}$$



Density of the induced surface charge, obtained using Coulomb's law:

$$\sigma(x) = \varepsilon_0 E_y(x) = -2\varepsilon_0 E_1 \cos\theta =$$

$$= -\frac{2\varepsilon_0}{4\pi\varepsilon_0}\cdot\frac{Q}{x^2+a^2}\cdot\frac{a}{\sqrt{x^2+a^2}} =$$

$$= -\frac{Q}{2\pi a^2}\left(\frac{a}{\sqrt{x^2+a^2}}\right)^3 = -\frac{Q}{2\pi a^2}\cos^3\theta$$

Total charge induced

Total (negative) charge induced on the surface:

$$Q_{\text{ind}} = \int_0^\infty \sigma(x)\cdot 2\pi x dx = -\frac{Q}{2\pi a^2}\int_0^\infty 2\pi x\left(\frac{a}{\sqrt{x^2+a^2}}\right)^3 dx$$

$$= -\frac{Qa}{2}\int_0^\infty \frac{d(x^2)}{(x^2+a^2)^{3/2}} = -\frac{Qa}{2}\int_{a^2}^\infty \frac{dt}{t^{3/2}} = \frac{Qa}{2}\cdot\frac{2}{\sqrt{t}}\bigg|_{a^2}^\infty = -Q$$

We can also see that $Q_{ind}=-Q$ using the Gauss law:
for a very large sphere, surface area $A \sim r^2$;
dipole field $E \sim 1/r^3$ (lecture 4); electric field flux through the surface:

$$\int_S \vec{E}d\vec{S} \sim \frac{1}{r} \to 0$$

Therefore, the net charge enclosed is $Q_{\text{ind}} + Q = 0$

Magnetostatics

[not discussed in the lecture]

Equations of magnetostatics (steady currents, j=const):

$$\nabla\vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{B} = \mu_0\vec{j}$$

Considering that $\nabla(\nabla \times \vec{A}) = 0$ for any vector field,
one can define the *vector potential*, $\vec{A}$, as follows:

$$\vec{B} = \nabla \times \vec{A}; \qquad \nabla\vec{A} = 0 \quad \text{“Coulomb gauge”}$$

Substituting into Ampere’s law,

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla\vec{A}) - \nabla^2\vec{A} = -\nabla^2\vec{A} = \mu_0\vec{j}$$

Finally, $\boxed{\nabla^2\vec{A} = -\mu_0\vec{j}}$

Equivalently, $\nabla^2 A_x = -\mu_0 j_x;\ \nabla^2 A_y = -\mu_0 j_y;\ \nabla^2 A_z = -\mu_0 j_z$

Analogous to the Poisson eqn of electrostatics: $\nabla^2\varphi = -\rho/\varepsilon_0$

This provides a general method for magnetostatic problems.

Summary

- Ideal conductors in electrostatic field:
  - ✓ the volume of the conductor is equipotential;
  - ✓ there are no volume charges ($\rho=0$);
  - ✓ in general, there are non-zero surface charges ($\sigma\neq 0$);
  - ✓ just above the surface, $E_n=\sigma/\varepsilon_0$ and $E_t=0$.

- Laplace equation for the electrostatic potential in the absence of free charges:

$$\nabla^2\varphi = 0$$

- The uniqueness theorem: the Laplace equation has a unique solution for any complete set of boundary conditions.

- The method of image charges: a tool for electrostatic field computations in the presence of conductors.