UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 4
Electric dipoles

- Electric dipole moment
- Electrostatic potential and electric field of a dipole
- Electric dipoles in an external electric field

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Previous lecture

- Ideal conductors in electrostatic field:
  - ✓ the volume of the conductor is equipotential;
  - ✓ there are no volume charges ($\rho=0$);
  - ✓ in general, there are non-zero surface charges ($\sigma\neq0$);
  - ✓ just above the surface, $E_n=\sigma/\varepsilon_0$ and $E_t=0$.

- Laplace equation for the electrostatic potential in the absence of free charges:

$$\nabla^2\varphi = 0$$

- The uniqueness theorem: the Laplace equation has a unique solution for any complete set of boundary conditions.

- The method of image charges: a tool for electrostatic field computations in the presence of conductors.

Electric dipole moment

A charge Q distributed over a volume V, near the origin of a reference frame:

$$Q = \int_V \rho(\vec{r}\,')dV'$$

By the principle of superposition, the electrostatic potential of this charge distribution:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV'$$

When observed from a large distance (r≫r'),

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = (r^2 + r'^2 - 2\vec{r}\vec{r}\,')^{-1/2} \approx \frac{1}{r}\left(1 - 2\vec{r}\vec{r}\,'/r^2\right)^{-1/2} \approx \frac{1}{r} + \frac{\vec{r}\vec{r}\,'}{r^3}$$

negligible

Coulomb term (dominates at r≫r' for Q≠0)

Therefore

$$\varphi(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}\,')dV'\left(\frac{1}{r} + \frac{\vec{r}\vec{r}\,'}{r^3}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\vec{p}\vec{r}}{r^3}\right)$$

Dipole term (the leading term for Q=0)

$$\vec{p} = \int_V \rho(\vec{r}\,')\vec{r}\,'dV'$$

is the *electric dipole moment* of the charge distribution [unit: C·m].

Electric dipoles

For a system of point charges, $\vec{p} = \sum_i q_i \vec{r}_i$

Electric dipole: a (neutral) system system with Q=0 and p≠0.
Examples: H_2O, C_2H_5OH, HCl molecules ($p \sim 10^{-29} - 10^{-30}$ C·m).

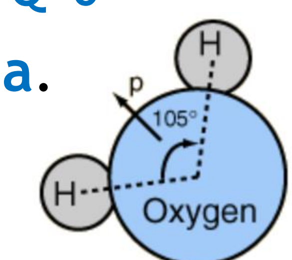
For Q=0, dipole moment *does not depend on the origin of coordinates*.

Proof for a system of point charges:
if the origin shifted by $\vec{a}$, coordinates become $\vec{r}_i{}' = \vec{r}_i - \vec{a}$

Dipole moment for the new choice of origin:

$$\vec{p}\,' = \sum_i q_i(\vec{r}_i - \vec{a}) = \sum_i q_i \vec{r}_i - \vec{a} \underbrace{\sum_i q_i}_{\text{Net charge, } Q=0} = \sum_i q_i \vec{r}_i = \vec{p}$$

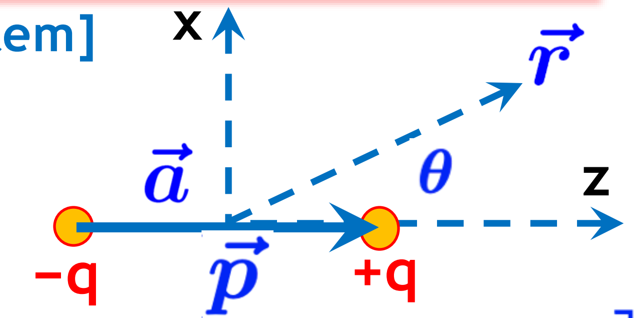
A simple (+q,−q) electric dipole:
two opposite point charges ±q, separated by a distance a.



Its dipole moment, using the above definition: $\vec{p} = q\vec{a}$

Potential for a (+q,−q) dipole

[not discussed in the lecture; a non-assessed problem]



Explicit computation for the simple dipole: using the principle of superposition, the electrostatic potential is

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(z-a/2)^2+x^2+y^2}} - \frac{q}{\sqrt{(z+a/2)^2+x^2+y^2}}\right]$$

At large distances ($|z| \gg a$),

$$(z \pm a/2)^2 + x^2 + y^2 \approx z^2 \pm za + x^2 + y^2 = r^2\left(1 \pm \frac{za}{r^2}\right)$$

Leading term in the Taylor expansion at large distances ($r \gg a$):

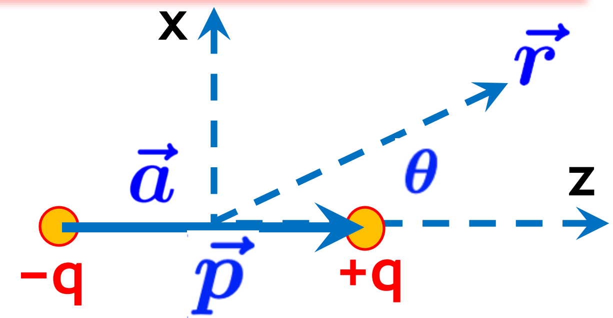
$$\varphi(\vec{r}) \approx \frac{q}{4\pi\varepsilon_0 r}\left[\left(1 - za/r^2\right)^{-1/2} - \left(1 + za/r^2\right)^{-1/2}\right] \approx$$

$$\approx \frac{q}{4\pi\varepsilon_0 r}\frac{za}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\vec{r}}{r^3}}$$

The dipole term, as expected

Dipole electric field

$$\vec{E} = -\nabla\varphi = -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{\vec{p}\vec{r}}{r^3}\right)$$



The **z**-component:

$$E_z = -\frac{p}{4\pi\varepsilon_0}\cdot\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{p}{4\pi\varepsilon_0}\cdot\frac{3z^2r - r^3}{r^6} = \frac{p}{4\pi\varepsilon_0}\cdot\frac{3\cos^2\theta - 1}{r^3}$$

Each of the (**x**,**y**) transverse components:

$$E_x = -\frac{p}{4\pi\varepsilon_0}\cdot\frac{\partial}{\partial x}\left(\frac{z}{r^3}\right) = \frac{p}{4\pi\varepsilon_0}\cdot\frac{3zx}{r^5}$$

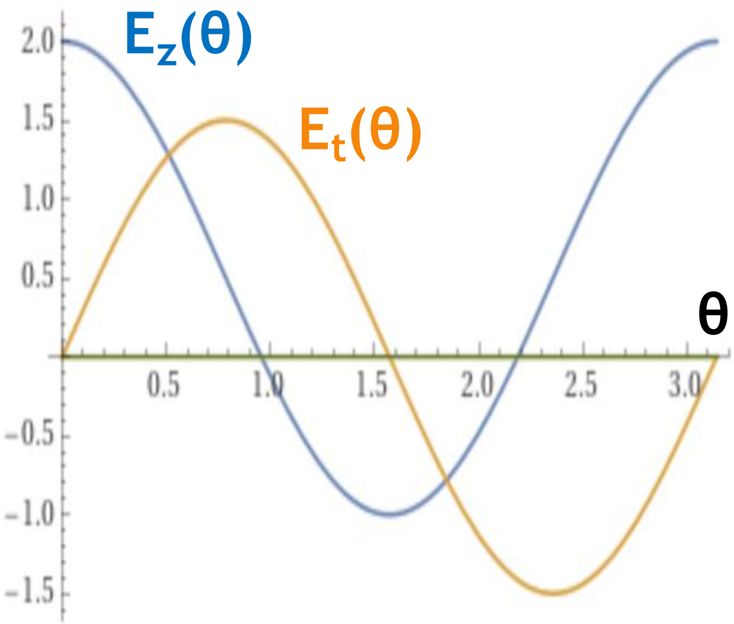
The total transverse component:

$$E_\perp = \sqrt{E_x^2 + E_y^2} = \frac{p}{4\pi\varepsilon_0}\cdot\frac{3z}{r^5}\sqrt{x^2+y^2} = \frac{p}{4\pi\varepsilon_0}\cdot\frac{3\sin\theta\cos\theta}{r^3}$$

At a fixed direction, dipole field falls off with distance as $\sim 1/r^3$.
Dipole field and potential are symmetric wrt the dipole axis.

Dipole electric field: illustration

The two components of the dipole field

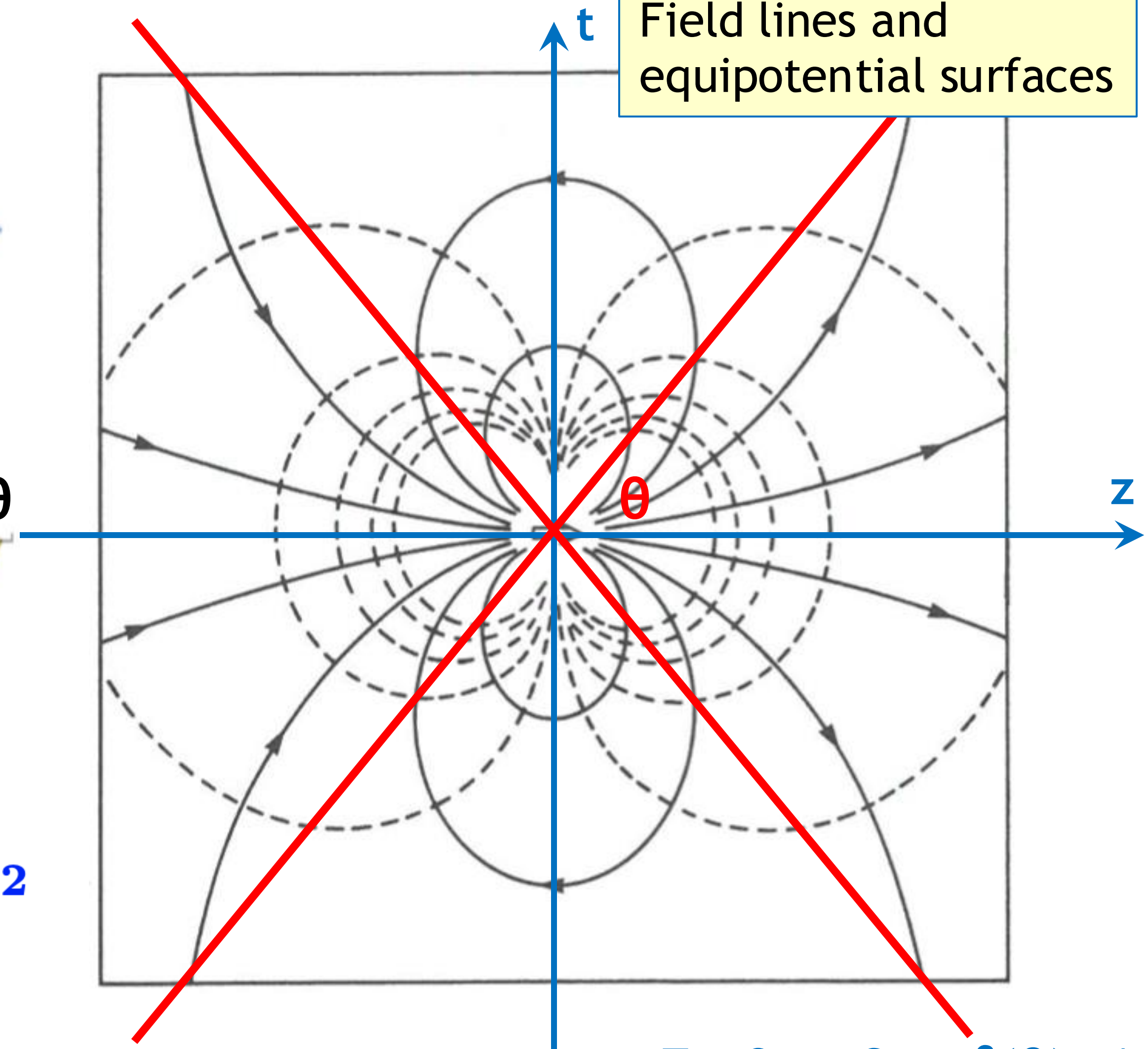


$$E_z|_{\theta=0} = -2E_z|_{\theta=\pi/2}$$

$$E_\perp|_{\theta=0;\pi/2;\pi} = 0$$

On the **z** axis: $E_z(z) = \dfrac{p}{2\pi\varepsilon_0|z|^3}$

Field lines and equipotential surfaces



$E_z=0$ at $3\cos^2(\theta)=1$, i.e. $\theta \approx \pm 1$ rad

Dipole in a uniform field

Force acting on a charge distribution in a uniform electric field E_0:

$$\vec{F} = \int_V \rho(\vec{r})\vec{E}_0 dV = \vec{E}_0 \int_V \rho(\vec{r}) dV = Q\vec{E}_0$$

Torque exerted on a charge distribution in a uniform field:

$$\vec{\tau} = \int_V \vec{r} \times \left(\rho(\vec{r}) dV \vec{E}_0\right) = \int_V (\rho(\vec{r})\vec{r} dV) \times \vec{E}_0 = \vec{p} \times \vec{E}_0$$

For a dipole (**Q=0**), there is *only* a torque,
as long as the dipole is not parallel to the external field.

Potential of a uniform field E_0: $\varphi(\vec{r}) = -\vec{E}_0\vec{r} + C$

Proof *(a non-assessed problem)*: $-\nabla\varphi(\vec{r}) = -\nabla(-\vec{E}_0\vec{r}) = \vec{E}_0$

Potential energy of a dipole in a uniform external field:

$$U = \int_V \varphi(\vec{r})\rho(\vec{r}) dV = \int_V (-\vec{E}_0\vec{r})\rho(\vec{r}) dV = -\vec{p} \cdot \vec{E}_0$$

Minimal energy: **p** in the direction of E_0.
The dipole moment tends to align with the external field.

Dipole in a non-uniform field

A non-uniform field, approximated by first-order terms in the Taylor expansion about the origin:

$$\vec{E}(\vec{r}) = \vec{E}_0 + x\frac{\partial \vec{E}}{\partial x} + y\frac{\partial \vec{E}}{\partial y} + z\frac{\partial \vec{E}}{\partial z} = \vec{E}_0 + (\vec{r}\nabla)\vec{E}$$

Force acting on a charge distribution placed at the origin:

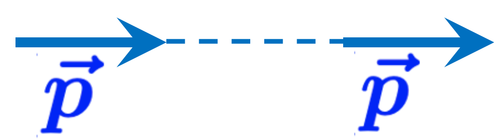
$$\vec{F} = \int_V \rho(\vec{r}) \left[\vec{E}_0 + (\vec{r}\nabla)\vec{E}\right] dV = Q\vec{E}_0 + (\vec{p}\nabla)\vec{E}$$

Force acting on a dipole (**Q=0**): $\boxed{\vec{F} = (\vec{p}\nabla)\vec{E}}$

Easier to understand considering individual components, e.g.

$$F_x = (\vec{p}\nabla)E_x = p_x\frac{\partial E_x}{\partial x} + p_y\frac{\partial E_x}{\partial y} + p_z\frac{\partial E_x}{\partial z}$$

Two coaxial point-like dipoles at a distance **x**:



$$F_x = p\frac{\partial}{\partial x}\left(\frac{p}{2\pi\varepsilon_0 x^3}\right) = -\frac{3p^2}{2\pi\varepsilon_0 x^4}$$

This is a specific type of van der Waals attraction.

Summary

❖ Electrostatic potential of a point-like electric dipole:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\,\vec{r}}{r^3}$$

❖ Electric dipoles in uniform fields: zero net force; non-zero torque; dipole moments tend to align along the field lines.

$$\vec{\tau} = \vec{p} \times \vec{E} \qquad U = -\vec{p} \cdot \vec{E}$$

❖ Electric dipoles in non-uniform fields: non-zero net force.
A dipole aligned with the field is attracted towards stronger field.

$$\vec{F} = (\vec{p}\,\nabla)\vec{E}$$

❖ Electrically neutral objects with non-zero dipole moments (**Q=0**, **p≠0**), e.g. water molecules, experience electrostatic forces.