UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 6
Maxwell's equations in free space

- The continuity equation
- Modification of Ampere's law for non-steady currents
- The displacement current
- Maxwell's equations of electrodynamics in free space

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Previous lectures

- Our current (still incomplete) understanding of the equations of electromagnetism *in free space*:

Gauss law (*universally valid*)

$$\nabla \vec{E} = \rho/\varepsilon_0$$

Faraday's law (*universally valid*)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Absence of magnetic poles (*universally valid*)

$$\nabla \vec{B} = 0$$

Ampere's law (*static field only*)

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

- Variation of magnetic field gives rise to electric field.
- Does variation of electric field give rise to magnetic field?

The continuity equation

Conservation of electric charge: any variation of charge within a fixed volume is due to charges flowing across the surface

$$\int_S \vec{j}\,d\vec{S} = -\frac{d}{dt}\int_V \rho\,dV$$

Unit on both sides: [C/s=A]

Divergence theorem: $$\int_S \vec{j}\,d\vec{S} = \int_V \nabla\vec{j}\,dV$$

Conservation of charge in differential form: (the *continuity equation*)

$$\boxed{\nabla\vec{j} = -\frac{\partial\rho}{\partial t}}$$

Let's take the divergence of both sides of Ampere's law:

$$\underbrace{0 = \nabla(\nabla\times B)}_{\text{T1 from lecture 2}} = \underbrace{\mu_0\nabla\vec{j} = -\mu_0\frac{\partial\rho}{\partial t}}_{\text{Continuity equation}}$$

Therefore, Ampere's law applies to **steady currents only** ($\partial\rho/\partial t = 0$), and **requires generalisation**

Generalised Ampere's law

To generalise Ampere's law for non-steady currents,
let's modify its right-hand side to make its divergence zero.

Gauss law: $\varepsilon_0 \nabla \vec{E} - \rho = 0$. Differentiation over **t**:

$$0 = \varepsilon_0 \nabla \left(\frac{\partial \vec{E}}{\partial t} \right) - \frac{\partial \rho}{\partial t} = \varepsilon_0 \nabla \left(\frac{\partial \vec{E}}{\partial t} \right) + \nabla \vec{j} = \nabla \left(\underbrace{\varepsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{Displacement current density}} + \vec{j} \right)$$

We have found a quantity $\left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{j} \right)$ which always has zero divergence and is therefore suitable for a modified Ampere's law.

The *Ampere–Maxwell law*: $\boxed{\nabla \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$ (*)

Not a proof: there and many ways of removing the contradiction. Any solenoidal field (i.e. such that $\nabla \vec{u} = 0$) can be added to the right-hand side of (*), still satisfying charge conservation.

The displacement current

$$\nabla \times \vec{B} = \mu_0(\vec{j}_C + \vec{j}_D) = \mu_0\vec{j}_C + \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}$$

j_C: conduction current density j_D: displacement current density

A changing **E**-field produces **B**-field:
Maxwell's decisive step towards electrodynamics (**1865**).

It took **30** years after Faraday's discovery of EM induction to postulate the displacement current as a source of magnetic field.

Alternating current in a copper wire

[conductivity $\sigma = 6\times10^{7}\ (\Omega\cdot m)^{-1}$]

Electric field: $E = E_0\sin(\omega t)$

Conduction current density: $j_C = \sigma E = \sigma E_0\sin(\omega t)$

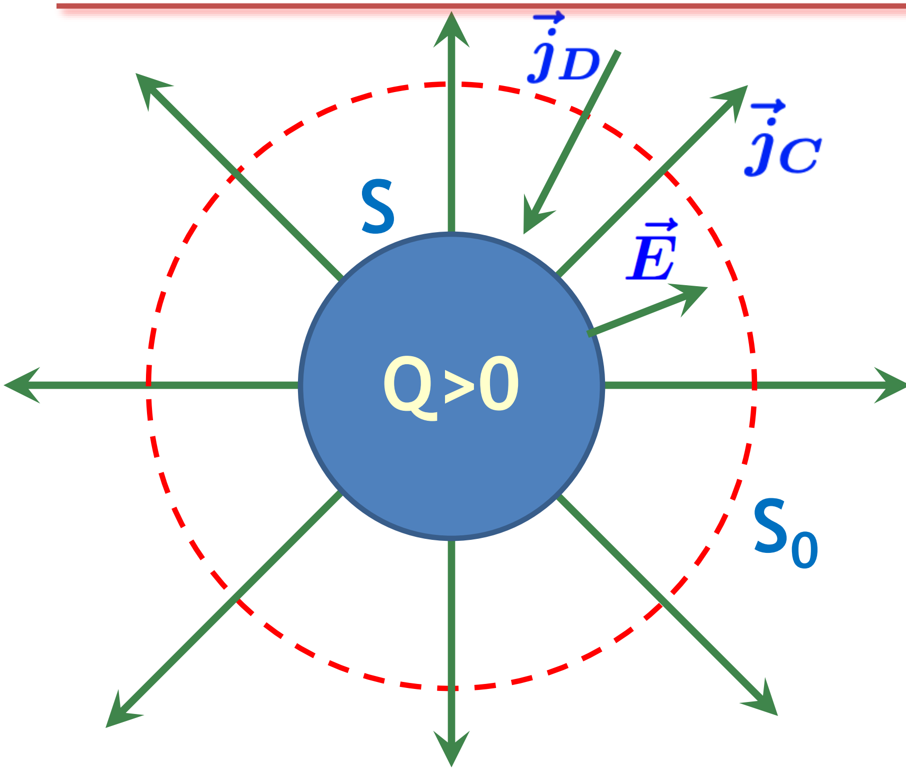
Displacement current density: $j_D = \varepsilon_0\partial E/\partial t = \varepsilon_0\omega E_0\cos(\omega t)$

The ratio of maximum max conduction to max displacement current:

$$j_C^{max} / j_D^{max} = \sigma E_0 / \varepsilon_0\omega E_0 \sim 10^{19}\ s^{-1}\ \omega^{-1} \gg 1,$$

i.e. j_D is negligible for all frequencies used in practice.

Example 1



A charged sphere discharging into external conductive medium.

By symmetry, $\vec{j}_C$ is directed radially.

What about the $\vec{B}$ field?

By symmetry, tangential component $B_t=0$, and radial component B_R is the same in each point of sphere S.

Assuming $B_R \neq 0$ leads to $\int_S \vec{B} d\vec{S} \neq 0$

We conclude that $B=0$.

Ampere's law leads to $j_C=0$.

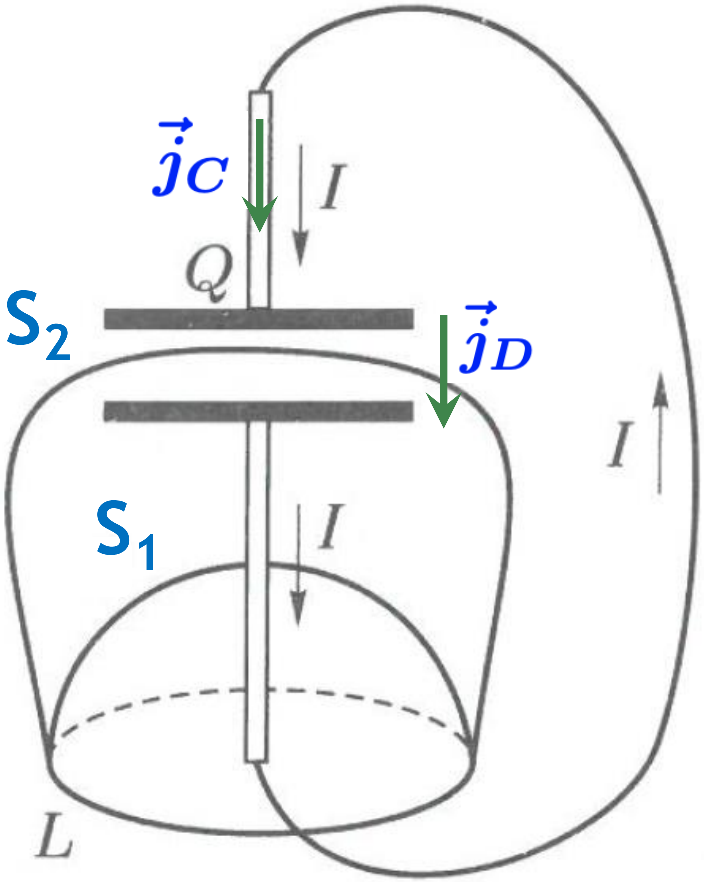
E-field at the surface of the sphere *(lecture 1)*: $E = Q/(4\pi\varepsilon_0 R^2)$

Conduction current at surface S_0 is balanced by displacement current:

$$I_D = I_C = \frac{dQ}{dt} = 4\pi\varepsilon_0 R^2 \frac{\partial E}{\partial t}$$

$$J_D = I_D/(4\pi R^2) = \varepsilon_0 \frac{\partial E}{\partial t}$$

Example 2



Discharge of a parallel-plate capacitor:

$$\oint_L \vec{B} d\vec{l} = \mu_0 \int_S (\vec{j}_C + \vec{j}_D) d\vec{S}$$

- For the surface S_1, only the conduction current I_C contributes.
- For the surface S_2, there is no conduction current, therefore $I_D = I_C$.

Let's check explicitly the equality $I_D = I_C$:

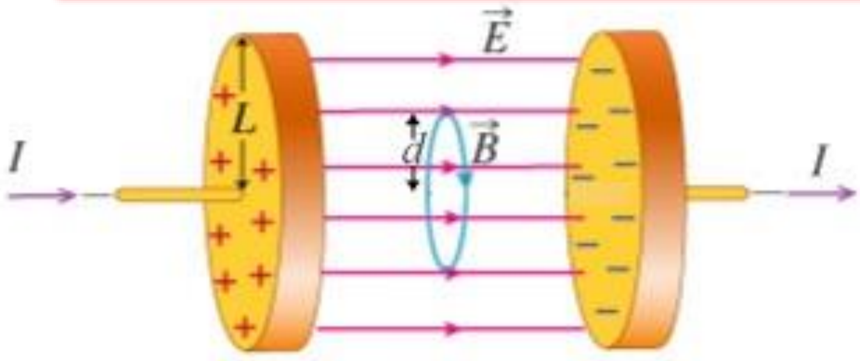
$$I_C = \frac{\partial Q}{\partial t} = A \frac{\partial \sigma}{\partial t} = A \varepsilon_0 \frac{\partial E}{\partial t}$$

Lecture 1: $E = \sigma / \varepsilon_0$

By definition, $I_D = A \varepsilon_0 \frac{\partial E}{\partial t} = I_C$

Example 3

[not discussed in the lecture]



A thin parallel plate capacitor with circular plates of radius **R** is being charged. Find the magnetic field **B(r)**.

Using axial symmetry, in the absence of conduction current (**j_C=0**),

$$\oint_L \vec{B} d\vec{l} = \mu_0 \int_S (\vec{j}_C + \vec{j}_D) d\vec{S} = \varepsilon_0 \mu_0 \int_S \frac{d\vec{E}}{dt}$$

Inside the capacitor (**r<R**),

$$2\pi r B = \varepsilon_0 \mu_0 \cdot \pi r^2 \frac{dE}{dt}$$

$$B(r) = \frac{1}{2} \varepsilon_0 \mu_0 r \frac{dE}{dt}$$

Outside the capacitor (**r≥R**),

$$2\pi r B = \varepsilon_0 \mu_0 \cdot \pi R^2 \frac{dE}{dt}$$

$$B(r) = \frac{1}{2} \varepsilon_0 \mu_0 \frac{R^2}{r} \frac{dE}{dt}$$

Maxwell's equations in free space

Laws of electrodynamics in free space: (remember them)

(M1) $$\nabla \vec{E} = \rho/\varepsilon_0$$

(M2) $$\nabla \vec{B} = 0$$

(M3) $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(M4) $$\nabla \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Displacement current density: $\vec{j}_D = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

The continuity equation, $\nabla \vec{j} = -\frac{\partial \rho}{\partial t}$, follows from (M1) and (M4).

Discussion and summary

- Maxwell's equations in free space (2 scalar + 2 vector) are equivalent to 8 scalar equations with 10 variables (E_x, E_y, E_z, B_x, B_y, B_z, j_x, j_y, j_z, ρ).

- Maxwell's equations must be complemented by the equations characterising the media. These will be discussed in lectures 7–12.

- The equations are not symmetric wrt electric and magnetic fields:
  - ✓ no magnetic poles as sources of **B** field **(M2)**;
  - ✓ no magnetic currents as sources of **E** field **(M3)**.

- For constant fields ($\partial \mathbf{E}/\partial t = \partial \mathbf{B}/\partial t = 0$), two independent groups,
  - ✓ electrostatics: $\mathrm{div}\, \mathbf{E} = \rho/\varepsilon_0$; $\mathrm{curl}\, \mathbf{E} = 0$;
  - ✓ magnetostatics: $\mathrm{div}\, \mathbf{B} = 0$; $\mathrm{curl}\, \mathbf{B} = \mu_0 \mathbf{j}$.