UNIVERSITY OF
BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 7
Dielectric materials

- Dielectric polarisation
- Surface and volume polarisation charges
- Electric susceptibility
- Microscopic theory of gaseous dielectrics

Prof. Evgueni Goudzovski • Physics West 212 • goudzovs@cern.ch

Previous lecture

Maxwell's equations in free space (remember them)

(M1) $$\nabla\vec{E} = \rho/\varepsilon_0$$

(M2) $$\nabla\vec{B} = 0$$

(M3) $$\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

(M4) $$\nabla \times \vec{B} = \mu_0\vec{j} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

Displacement current density: $\vec{j}_D = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$

The continuity equation, $\nabla\vec{j} = -\frac{\partial\rho}{\partial t}$, follows from (M1) and (M4).

Conductors vs dielectrics

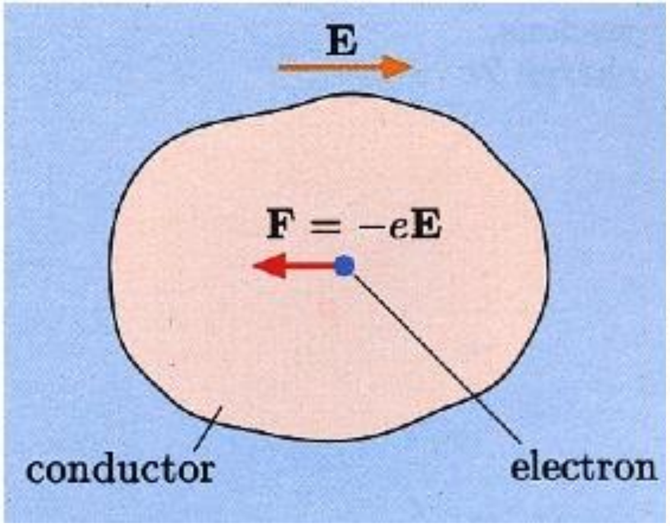
Conductors

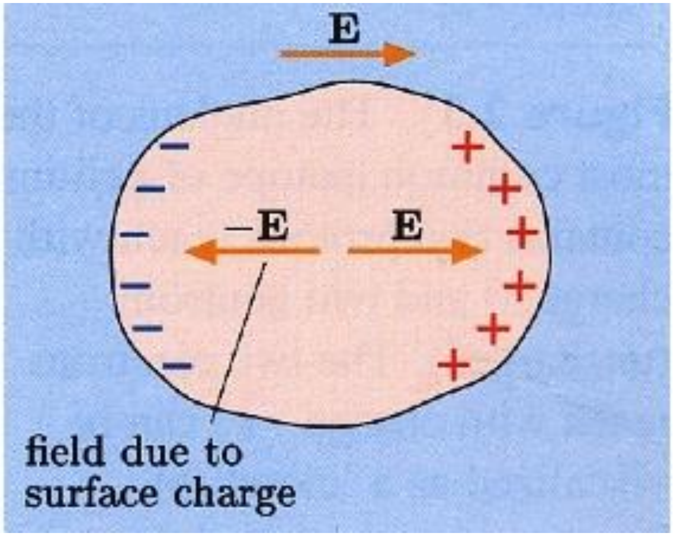
Free charges
Currents can flow: $j \neq 0$
Large conductivity σ [Ohm's law: $j = \sigma E$]
In the static limit, $\rho = E = 0$ inside
Surface+volume are equipotential
Free charges on surface only
E is perpendicular to surface

Dielectrics (insulators)

Bound charges
No currents flow: $j = 0$
Conductivity $\sigma = 0$
In the static limit, $E \neq 0$ inside
Surface+volume are **not** equipotential
Bound charges can be anywhere
Any **E** allowed at surface

Free charges in conductors:





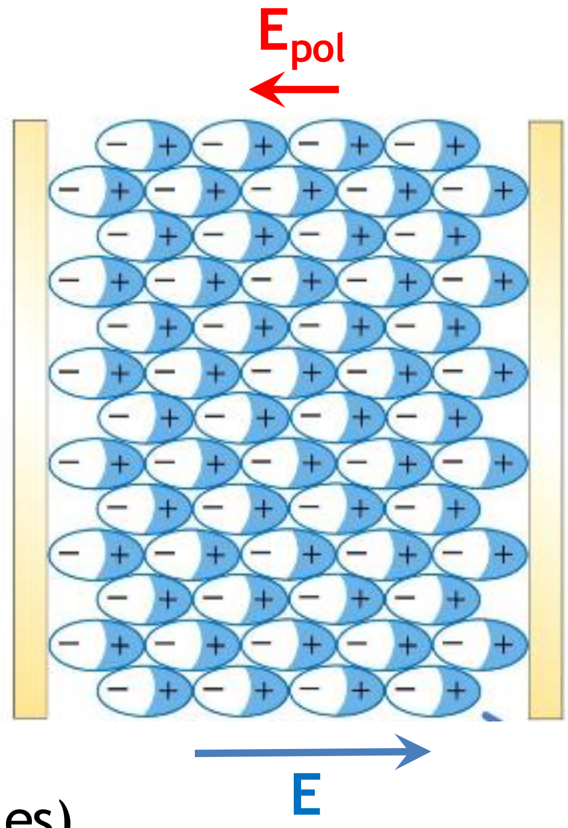
Dielectric polarisation

Main sources of **dielectric polarisation**:

1) *Electronic polarisation*: deformation of electronic clouds, alignment of *induced dipole moments* of the atoms along external **E**-field.

2) Orientation of *permanent molecular dipole moments* along the **E**-field (for polar molecules).

Polarisation *reduces* the field in the dielectric.



Polarisation vector: dipole moment induced per unit volume,

$$\vec{P} = \frac{1}{\Delta V} \sum_{\Delta V} \vec{p}_i$$

(sum over atoms/molecules)
[C/m^2]

Dielectric materials are attracted into regions of strong **E**-field:
dipole momenta (**p**) are aligned with **E**-field;
force on a dipole in non-uniform field is **F_z = p·dE/dz** *(lecture 4)*.

Surface polarisation charges

Polarisation gives rise to bound *surface charges*.

Dipole moment of a slice
(using the definition of the dipole moment):

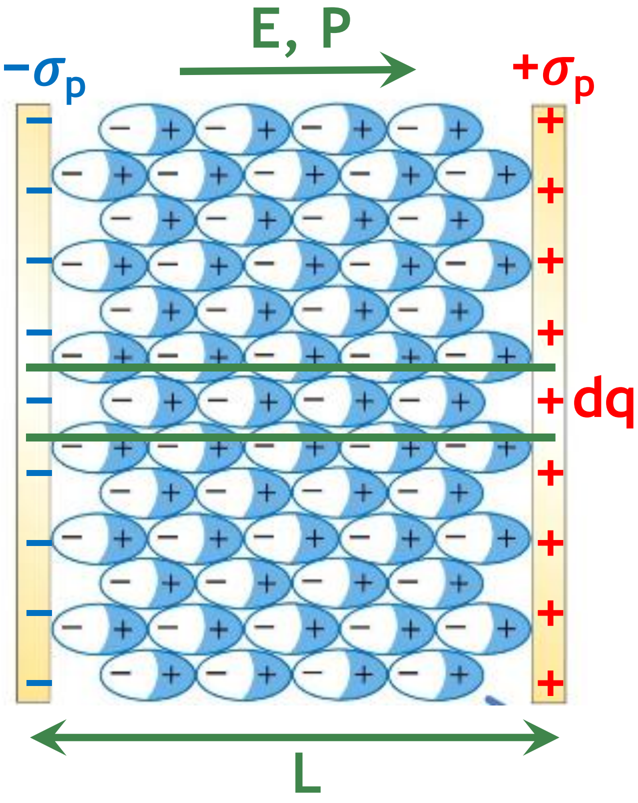
$$dp = dq \cdot L = \sigma_\mathrm{p} dS \cdot L = \sigma_\mathrm{p} dV$$

Total dipole moment of the sample:

$$p_\mathrm{total} = \int_V dp = \int_V \sigma_\mathrm{p} dV = \sigma_\mathrm{p} V$$

Polarisation (using the definition):

$$P = p_\mathrm{total}/V = \sigma_\mathrm{p}$$



More generally, for a surface not perpendicular to the polarisation vector,

$$\boxed{\sigma_\mathrm{p} = \vec{P} \cdot \vec{n}}$$

$\vec{n}$: unit normal vector pointing outwards.

No surface charges on the top and bottom surfaces in the sketch.

Volume polarisation charges

Elementary surface polarisation charge: $dQ = \vec{P} \cdot d\vec{S}$

Apply this to any volume **V** inside the dielectric to compute the *volume polarisation charge* density ρ_p:

$$\underbrace{Q_p}_{\text{Charge displaced } \textit{into} \text{ the volume V}} = \int_V \rho_p dV = \underbrace{-\int_S \vec{P} d\vec{S} = -\int_V \nabla \vec{P} dV}_{\text{Divergence theorem (}\textit{lecture 2}\text{)}}$$

Therefore $\boxed{\rho_p = -\nabla \vec{P}}$ (no volume charges for uniform polarisation)

If electric field **E(t)** varies with time, *polarisation current can* flow:

$$\underbrace{\nabla \vec{j}_p = -\frac{\partial \rho_p}{\partial t}}_{\text{Continuity equation (}\textit{lecture 6}\text{)}} = \frac{\partial}{\partial t} \nabla \vec{P}$$

, therefore $\boxed{\vec{j}_p = \frac{\partial \vec{P}}{\partial t}}$.

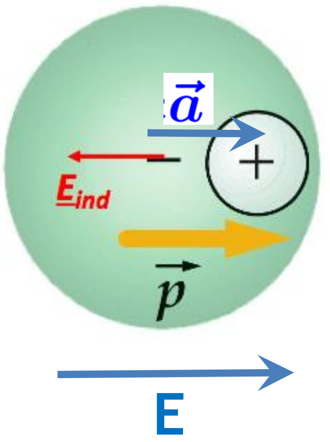
The distributions σ_p and ρ_p represent the effect of polarisation via equivalent macroscopic charge distributions.

Electric susceptibility

Consider a linear, isotropic and homogeneous (“LIH”) dielectric.

The *electric susceptibility* χ_E is defined as the (dimensionless) constant of proportionality between polarization and field:

$$\vec{P} = \chi_E \cdot \varepsilon_0 \cdot \vec{E}$$



Should we expect linearity?

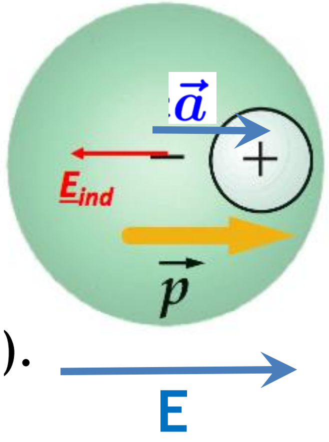
1) Electronic polarisation: electric field **E** causes displacement of electron cloud density wrt nucleus, therefore each atom becomes polarised.

Typical field “seen” by electrons:
(huge, greatly exceeds any external field)

$$E = \frac{e}{4\pi\varepsilon_0 r_{\text{Bohr}}^2} \sim 10^{11}\ \text{V/m}$$

Therefore the displacement is $a \ll r_{\text{Bohr}}$ (typically ~1 fm), and depends *linearly* on **E**.

Induced dipole moment



Atomic/molecular polarisability α [m^3]: property of atom/molecule, defined via the induced dipole moment:

$$\vec{p} = \alpha\varepsilon_0\vec{E}_{\mathrm{local}}$$

A crude estimate: $\alpha \approx 4\pi r^3_{\mathrm{atom}}$ (see non-assessed problems).

Electric susceptibility for a **gaseous non-polar dielectric**:

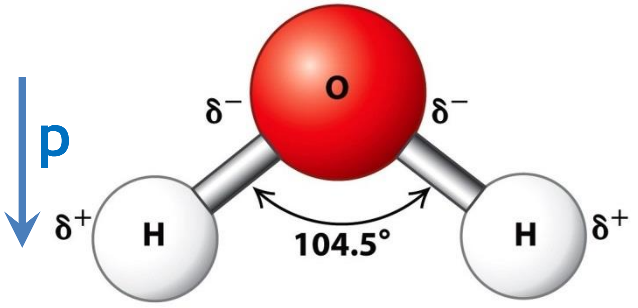
$$\chi_E = \frac{P}{\varepsilon_0 E} = \frac{np}{\varepsilon_0 E} = \frac{n\alpha\varepsilon_0 E}{\varepsilon_0 E} = n\alpha$$

α: molecular polarisability [m^3];
n: density of molecules [m^{-3}];
p: induced dipole momentum of a single molecule [C·m].

- Susceptibility depends on the density only; $\chi_E \sim 10^{-4}$ for gases at STP.
- Susceptibility does not depend on temperature at fixed density (assuming molecules do not raise to excites stated or dissociate).

Polar molecules

2) Polar molecules: **non-zero dipole moment**
Example: water (**H_2O**).



Typical molecular dipole moment:

$$p \approx 1.6 \times 10^{-19}\ \mathrm{C} \cdot 10^{-10}\ \mathrm{m} \sim 10^{-29}\ \mathrm{Cm}$$

Even for a very strong field (**$E=10^6$ V/m**),
the work required to revert the direction of **p** *(lecture 4)*

$$W = 2pE = 2 \times 10^{-29}\ \mathrm{Cm} \cdot 10^{6}\ \mathrm{V/m} \sim 10^{-4}\ \mathrm{eV}$$

is much smaller than thermal motion energy at room temperature:

$$\frac{3}{2}kT \approx 0.04\ \mathrm{eV}$$

Weak alignment of the dipoles, no saturation, *linearity*.

Orientation of dipole moments

In the absence of electric field, the density of dipole moments is isotropic:

$$dN = \frac{d\Omega}{4\pi} = \frac{2\pi \sin\theta d\theta}{4\pi} = \frac{1}{2}\sin\theta d\theta$$

In the presence of an external electric field,

θ: angle between dipole moment and field direction

$$dN = C \cdot \underbrace{\exp\left(-\frac{U}{kT}\right)}_{\text{Boltzmann factor}} \cdot 2\pi \sin\theta d\theta$$

$$\approx C \cdot \left(1 - \frac{U}{kT}\right) \cdot 2\pi \sin\theta d\theta$$

$$= C \cdot \left(1 + \frac{pE\cos\theta}{kT}\right) \cdot 2\pi \sin\theta d\theta$$

Normalisation: $\int_0^\pi dN = 4\pi C = 1$, therefore $C = \frac{1}{4\pi}$

Polarisation and susceptibility

Polarisation = dipole moment of a unit volume:

$$P = n \int_0^\pi \underbrace{\frac{1}{2}\sin\theta\left(1+\frac{pE\cos\theta}{kT}\right)}_{\text{Density of dipole moments}} \cdot \underbrace{p\cos\theta}_{\text{Component along the field direction}} \cdot d\theta$$

$$= \frac{np^2E}{2kT}\int_0^\pi \sin\theta\cos^2\theta \cdot d\theta = -\frac{np^2E}{2kT}\int_0^\pi \cos^2\theta d(\cos\theta) = \frac{np^2E}{3kT}$$

Net polarisation of gaseous polar dielectric:

$$\vec{P} = \left(n\alpha\varepsilon_0 + \frac{np^2}{3kT}\right)\vec{E}$$

Electric susceptibility for a gaseous polar dielectric, $\chi_E = P/(\varepsilon_0 E)$:

$$\chi_E = n\left(\alpha + \frac{p^2}{3\varepsilon_0 kT}\right)$$

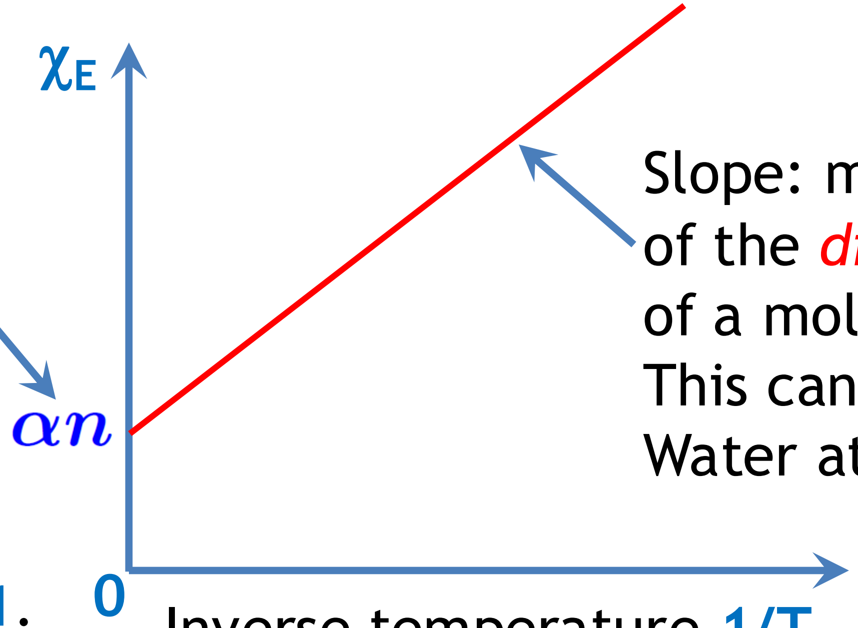
Susceptibility vs temperature

For **gaseous polar dielectrics**,

$$\chi_E = n\left(\alpha + \frac{p^2}{3\varepsilon_0 kT}\right)$$

At a **fixed density** (i.e. $n=const$):

Intercept: susceptibility due to *electronic polarisation*, direct measurement of the *molecular polarisability*.



Slope: measurement of the *dipole moment* of a molecule.
This can generate $\chi_E \gg 1$.
Water at $T=300K$: $\chi_E=80$.

Monoatomic gases at STP: $\alpha n \sim 10^{-4}$.
Solid dieletcrics: $\alpha n \sim 1$.

Liquid and solid dielectrics

[Not discussed in the lecture]

Liquids and solids have much higher densities (n) than gases:

- Much larger electronic polarisation term, αn.
- The field on each individual atom is influenced by polarisation of the neighbouring atoms.
- As a result, further enhancement of electronic polarisation. *Clausius–Mossotti relation* for non-polar liquid dielectrics:

$$\chi_E = \frac{n\alpha}{1 - \frac{1}{3} n\alpha}$$

- Solids (including piezoelectricity, pyroelectricity, ferroelectricity) and polar liquids are not discussed in this course.

Summary

- Polarisation of LIH dielectric materials, characterised by the electric susceptibility χ_E:

$$\vec{P} = \frac{1}{\Delta V} \sum_{\Delta V} \vec{p_i} = \chi_E \cdot \varepsilon_0 \cdot \vec{E}$$

- Volume distribution of polarisation **P** is equivalent to the following distribution of surface and volume charges:

$$\sigma_\mathrm{p} = \vec{P} \cdot \vec{n} \quad \text{and} \quad \rho_\mathrm{p} = -\nabla \vec{P}$$

- Electric susceptibility for gaseous dielectrics:

$$\chi_E = n \left(\alpha + \frac{p^2}{3\varepsilon_0 k T} \right)$$

The two terms are due to *electronic polarisation* and the *alignment of permanent dipole moments* ***p*** of molecules.