UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 8
Electric field in dielectric materials

- The electric displacement field **D**; relative permittivity
- Gauss law for the **D** field
- Boundary conditions for the **E** and **D** fields
- Energy density of the electric field

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Previous lecture

- Polarisation of LIH dielectric materials, characterised by the electric susceptibility χ_E:

$$\vec{P} = \frac{1}{\Delta V}\sum_{\Delta V}\vec{p}_i = \chi_E \cdot \varepsilon_0 \cdot \vec{E}$$

- Volume distribution of polarisation **P** is equivalent to the following distribution of surface and volume charges:

$$\sigma_p = \vec{P}\cdot\vec{n} \quad \text{and} \quad \rho_p = -\nabla\vec{P}$$

- Electric susceptibility for gaseous dielectrics:

$$\chi_E = n\left(\alpha + \frac{p^2}{3\varepsilon_0 kT}\right)$$

The two terms are due to *electronic polarisation* and the *alignment of permanent dipole moments* ***p*** of molecules.

The electric displacement field

Dielectric materials in non-uniform fields: *(lecture 7)*

induced *volume polarisation charges* with density $\boldsymbol{\rho_\mathrm{p} = -\nabla \vec{P}}$

In general, dielectrics also carry *free charges* with a density $\boldsymbol{\rho_\mathrm{f}}$

Total charge density: $\boldsymbol{\rho = \rho_\mathrm{f} + \rho_\mathrm{p}}$

Gauss law: $\boldsymbol{\nabla \vec{E} = \frac{\rho}{\varepsilon_0} = \frac{1}{\varepsilon_0}(\rho_\mathrm{f} + \rho_\mathrm{p}) = \frac{1}{\varepsilon_0}\left(\rho_\mathrm{f} - \nabla \vec{P}\right)}$

Therefore $\boldsymbol{\varepsilon_0 \nabla \vec{E} + \nabla \vec{P} = \rho_\mathrm{f}}$ and $\boldsymbol{\nabla(\varepsilon_0 \vec{E} + \vec{P}) = \rho_\mathrm{f}}$

Let's define the *electric displacement field*:

$$\boldsymbol{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

Gauss law for the D field

Gauss law for the electric displacement field: $\nabla \vec{D} = \rho_{\mathrm{f}}$

This is a generalised version of Maxwell's equation (M1): $\nabla \vec{E} = \rho / \varepsilon_0$

In the integral form *(lectures 1,2)*: $\int_S \vec{D} d\vec{S} = \int_V \rho_{\mathrm{f}} dV$

Using the definition of electric susceptibility, $\vec{P} = \chi_E \cdot \varepsilon_0 \cdot \vec{E}$,

$$\vec{D} = (1 + \chi_E) \cdot \varepsilon_0 \vec{E} = \varepsilon \varepsilon_0 \vec{E}$$

where $\varepsilon = 1 + \chi_E$ is the *relative permittivity* (aka the *dielectric constant*) of the medium.

Gauss law for uniform ε: $\nabla \vec{E} = \rho_{\mathrm{f}} / (\varepsilon_0 \varepsilon)$

- The **E**-field *decreases* by a factor of $\varepsilon > 1$ relative to the field produced by the same distribution of free charges in free space.
- Ideal conductors: $\vec{E} = 0$ *(lecture 3)*, therefore $\varepsilon \to \infty$.

Physical meaning

Assumptions behind the idea of a constant ε for a dielectric:

- **Linearity**: **P** is proportional to **E**.
  This breaks down at high electric fields.
- **Isotropy**: polarisation vector **P** is collinear to **E**.
  NB: some crystals are anisotropic,
  particularly under mechanical stress (the piezoelectric effect).
- **Homogeneity**: no dependence on the position within material.

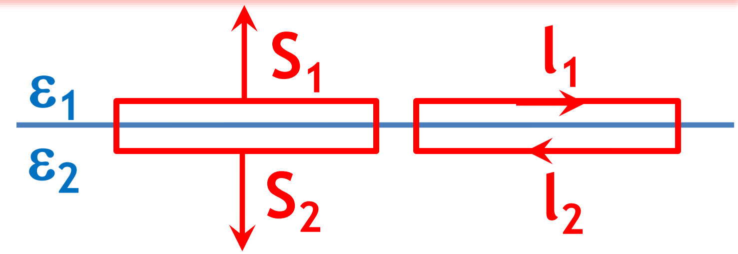
Most insulators satisfy these conditions in practice
(and are known as "**LIH dielectrics**").

Physical meaning:

- Electric field **E** [V/m]: *force* acting on unit charge.
- Polarisation **P** [C/m^2]: induced *dipole moment* per unit volume,
  a local property of matter.
- Electric displacement **D** [C/m^2]: introduced to facilitate
  field calculations in the presence of dielectrics
  (polarisation charges do not enter the Gauss law for **D**-field).

Boundary conditions for E and D

Boundary of two dielectrics
with relative permittivities ε_1 and ε_2



Gauss law for a very thin "pillbox" cylinder (no free surface charges):

$$\int_S \vec{D} d\vec{S} = 0 \quad \text{therefore} \quad \vec{D}_1 \vec{S}_1 + \vec{D}_2 \vec{S}_2 = 0$$

Using $\vec{S}_1 = -\vec{S}_2$, we obtain $\vec{D}_1 \vec{S}_1 = \vec{D}_2 \vec{S}_1$ and $\boxed{D_{1n} = D_{2n}}$

More generally, the discontinuity of D_n equals to surface density of *free charges*: $\boxed{D_{1n} - D_{2n} = \sigma}$

A very thin rectangular loop: infinitely small area, $\int_S \vec{B} d\vec{S} = 0$. Therefore the Faraday's law,

$$\oint_L \vec{E} d\vec{l} = \vec{E}_1 \vec{l}_1 + \vec{E}_2 \vec{l}_2 = 0, \quad \text{leads to} \quad \boxed{E_{1t} = E_{2t}}$$

Field lines at boundaries

D_n is continuous:

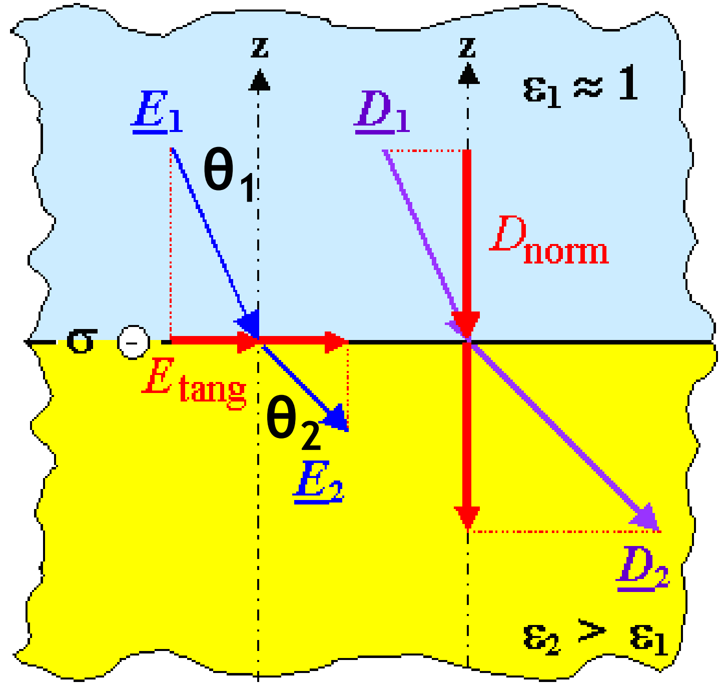
$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$$

E_t is continuous:

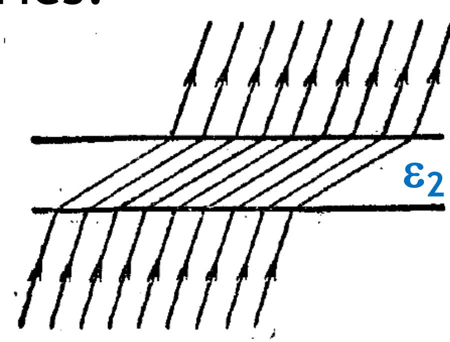
$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

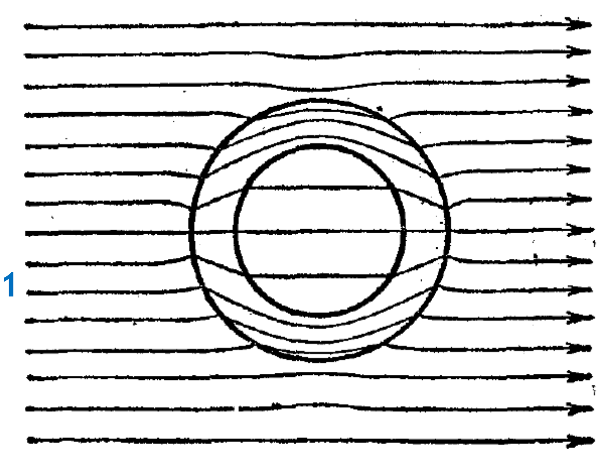
Therefore $$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

(similar to Snell's law in optics – not by accident!)

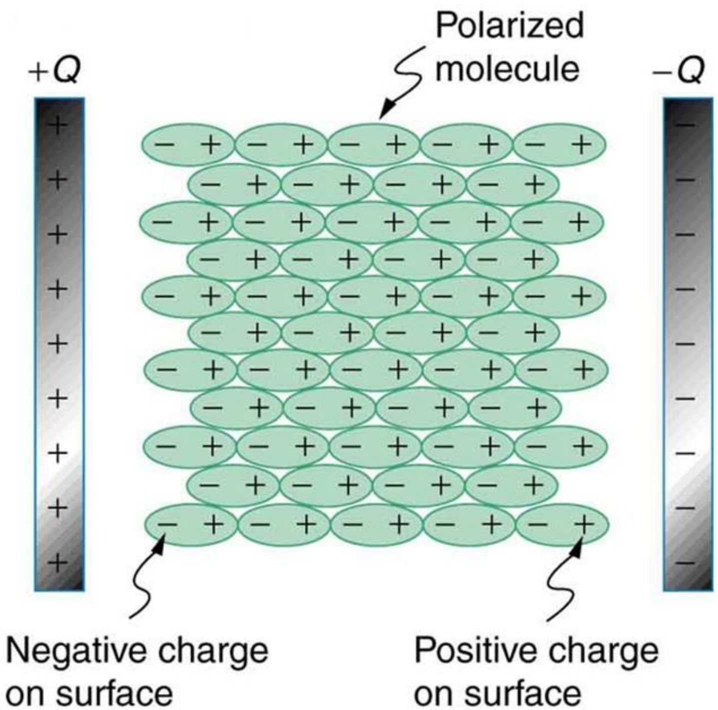


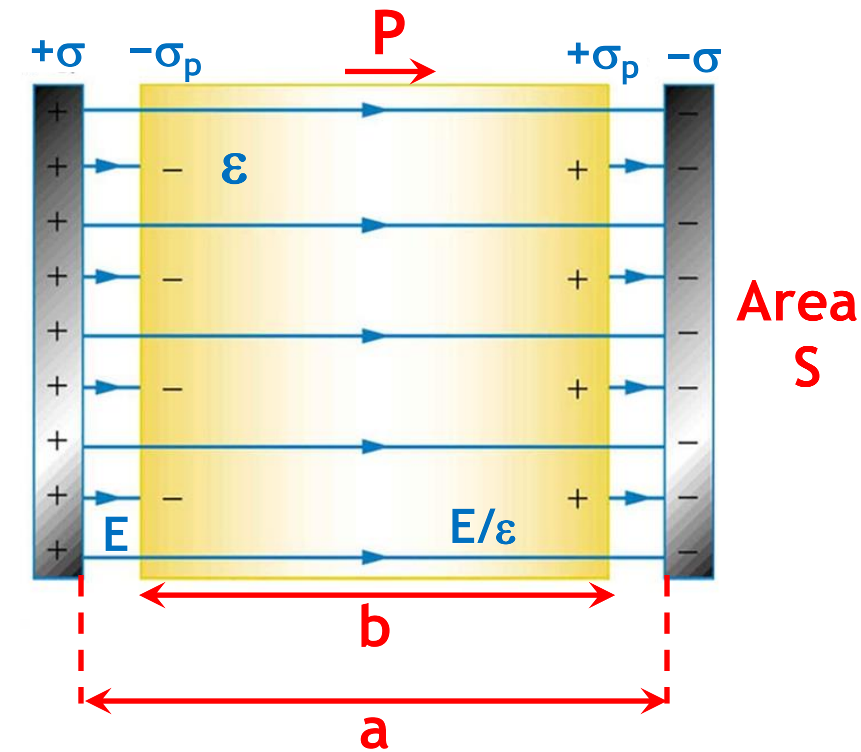
- ❖ Electrostatic field is partially shielded in cavities in high-ε dielectrics.
- ❖ Cf. complete shielding in a hollow conductor (Faraday's cage, *lecture 3*)





A dielectric slab in a capacitor (1)





- The displacement field **D** is uniform (because D_n is continuous).
- From Gauss law, $D = \sigma$ everywhere inside the capacitor.
- Therefore $E = D/(\varepsilon\varepsilon_0)$ is *reduced* by a factor of ε in the dielectric.
- From Gauss law, surface polarisation charge: $\sigma_p = \sigma(\varepsilon-1)/\varepsilon$.

A dielectric slab in a capacitor (2)

Potential difference between the two plates:

$$V = \int_0^a E dx = \int_0^a \frac{D}{\varepsilon_0 \varepsilon} dx = \frac{D}{\varepsilon_0}(a-b) + \frac{D}{\varepsilon_0 \varepsilon} b = \frac{\sigma}{\varepsilon_0}(a-b) + \frac{\sigma}{\varepsilon_0 \varepsilon} b$$

For **b=0** (no dielectric slab), $V_1 = \dfrac{\sigma a}{\varepsilon_0}$

For **b=a** (slab occupying all space), $V_2 = \dfrac{\sigma a}{\varepsilon_0 \varepsilon}$: *reduces* by a factor **ε**.

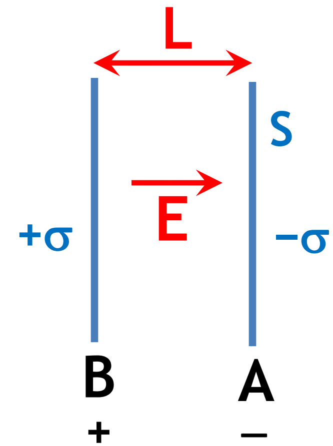
Capacitance in the two cases:

$$C_1 = \frac{Q}{V_1} = \frac{\sigma S}{V_1} = \frac{\varepsilon_0 S}{a}$$

$$C_2 = \frac{Q}{V_2} = \frac{\sigma S}{V_2} = \frac{\varepsilon_0 \varepsilon S}{a}$$: *increases* by a factor **ε**.

Energy density of the E-field (1)

Work required to move a charge $\delta q = S\delta\sigma$ from plane **A** to plane **B** in a parallel-plate capacitor filled with a dielectric material:



$$\delta W = EL \cdot \delta q = ELS \cdot \delta\sigma = VE \cdot \delta\sigma$$

(S: area; σ: surface charge density; V: volume)

Inside the capacitor, $D = \sigma$, therefore $\delta D = \delta\sigma$

Finally, $\delta W = VE \cdot \delta D$

The increment in *energy density* [J/m³]: $\delta w = \frac{\delta W}{V} = E\delta D$

More generally (also for anisotropic dielectrics), $\delta w = \vec{E}\delta\vec{D}$

Energy density of the E-field (2)

In the assumption $\vec{D} = \varepsilon\varepsilon_0\vec{E}$ (i.e. for LIH dielectrics),
the *energy density* is

$$w = \int_0^D dw = \int_0^D EdD = \varepsilon_0\varepsilon \int_0^E EdE = \frac{1}{2}\varepsilon_0\varepsilon E^2 = \frac{1}{2}ED$$

of which $\frac{1}{2}\varepsilon_0(\varepsilon - 1)E^2$ is due to polarisation of the dielectric.

The polarisation contribution often dominates (e.g. water: **ε=81**).

More generally (also for anisotropic dielectrics), $w = \frac{1}{2}\vec{E}\vec{D}$

In electrostatics, field energy can be interpreted as the potential energy of electric charges (assuming action at a distance).

Electrodynamic interpretation: *energy is localised in space*, and is carried by the electric field.

Summary

- The electric displacement field is defined as

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

- Gauss law for the electric displacement field:

$$\nabla \vec{D} = \rho_{\mathrm{f}} \quad \text{or equivalently} \quad \int_S \vec{D} d\vec{S} = \int_V \rho_{\mathrm{f}} dV$$

- For linear, isotropic and homogeneous (e.g. most) dielectrics,

$$\vec{D} = (1 + \chi_E) \cdot \varepsilon_0 \vec{E} = \varepsilon \varepsilon_0 \vec{E} \qquad (\varepsilon > 1)$$

- Boundary conditions (no free surface charges):

$D_{1n} = D_{2n}$ and $E_{1t} = E_{2t}$, leading to $\dfrac{\tan \theta_1}{\tan \theta_2} = \dfrac{\varepsilon_1}{\varepsilon_2}$

- Energy density of the electric field: $w = \dfrac{1}{2} \vec{E} \vec{D}$