UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 9
Diamagnetism and paramagnetism

- Magnetic properties of atoms
- Magnetisation and magnetic susceptibility
- Microscopic theory of diamagnetism and paramagnetism

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Previous lecture

❖ The electric displacement field is defined as

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

❖ Gauss law for the electric displacement field:

$$\nabla \vec{D} = \rho_{\mathrm{f}} \quad \text{or equivalently} \quad \int_S \vec{D} d\vec{S} = \int_V \rho_{\mathrm{f}} dV$$

❖ For linear, isotropic and homogeneous (e.g. most) dielectrics,

$$\vec{D} = (1 + \chi_E) \cdot \varepsilon_0 \vec{E} = \varepsilon \varepsilon_0 \vec{E} \qquad (\varepsilon > 1)$$

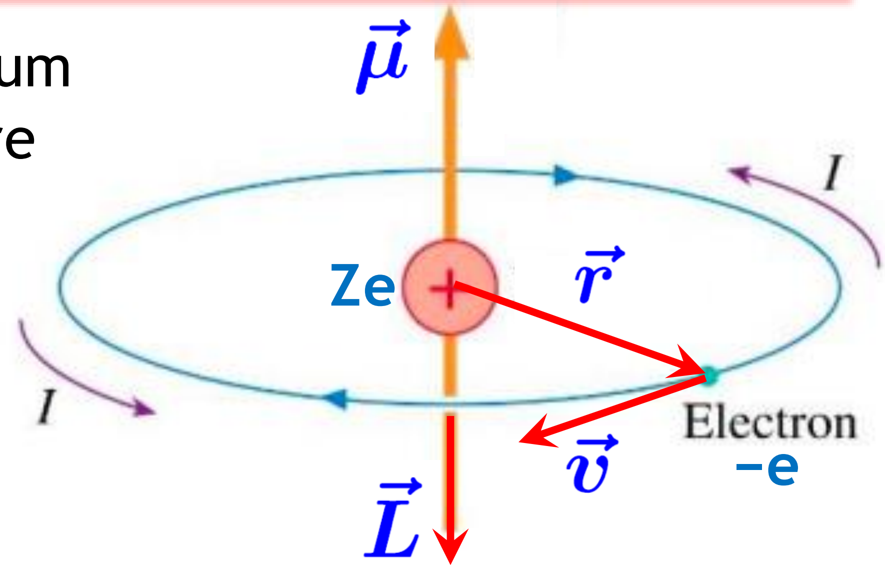
❖ Boundary conditions (no free surface charges):

$D_{1n} = D_{2n}$ and $E_{1t} = E_{2t}$, leading to $$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

❖ Energy density of the electric field: $w = \frac{1}{2} \vec{E} \vec{D}$

Magnetic properties of atoms (1)

Magnetism is a fundamentally quantum effect: magnetisation phenomena are absent in classical physics.



Bohr's semiclassical model

Angular momentum of the electron

$$\vec{L} = m\vec{r} \times \vec{v}$$

Magnetic dipole moment of a current loop *(lecture 5)*

$$\mu = IS = -\frac{e}{T} \cdot \pi r^2 = -e\frac{v}{2\pi r} \cdot \pi r^2 = -\frac{evr}{2}$$

(**T**: rotation period; **S**: area of loop; **e>0**: elementary charge)

Gyromagnetic ratio of electron orbital motion: $\Gamma = \frac{\mu}{L} = -\frac{e}{2m}$

Magnetic properties of atoms (2)

Quantisation of angular momentum component: $L = n\hbar, \quad n \in \mathbb{N}$

Quantisation of the orbital magnetic moment of electron:

$$\mu = \frac{e\hbar}{2m_e} \cdot n$$

Bohr magneton: $\mu_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24}\ \mathrm{A \cdot m^2}$

Electron spin (intrinsic angular momentum): $S = \pm\hbar/2$

The associated magnetic moment is $\mu_{spin} \approx \mu_B$, therefore
the gyromagnetic ratio is ~2 times larger than for orbital motion:

$$\Gamma_{spin} \approx -e/m.$$

Types of magnetism

Magnetism is linked to the charged constituents in materials:

1) orbital angular momenta of electrons;
2) spin angular momenta of electrons;
3) spin angular momenta of nuclei (a much smaller contribution).

In external magnetic field, materials acquire non-zero *magnetisation* $\vec{M}$, i.e. induced magnetic dipole moment per unit volume,

$$\vec{M} = \frac{1}{\Delta V} \sum_{\Delta V} \vec{\mu}_i$$ (sum over molecules; unit: [A/m])

For linear, isotropic and homogeneous materials, the *magnetic susceptibility*, χ_B, is defined as

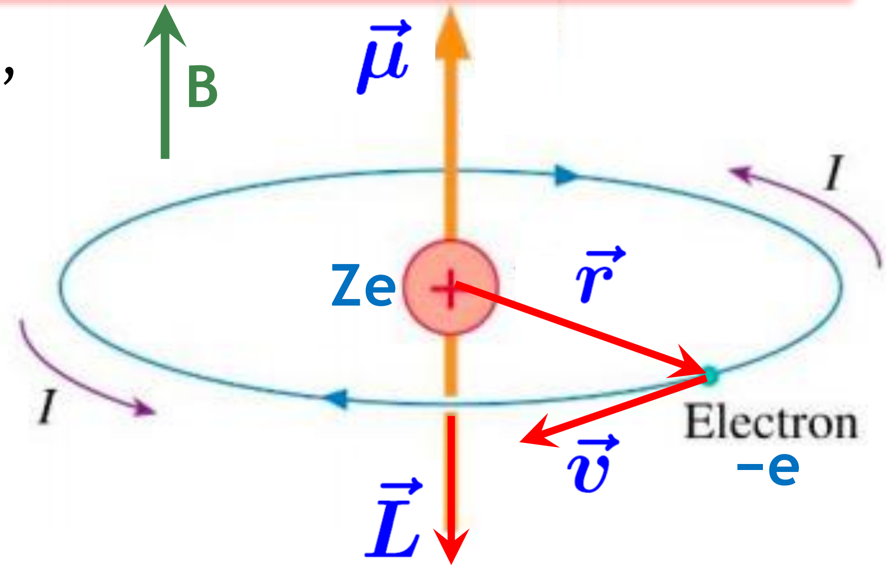
$$\vec{M} = \chi_B \frac{\vec{B}}{\mu_0}$$

Three main types of magnetic materials.

- *Diamagnetic*: **M** is opposite to external **B**; $\chi_B<0$ (and $|\chi_B| \ll 1$).
- *Paramagnetic*: **M** is parallel to external **B**; $\chi_B>0$ (and $|\chi_B| \ll 1$).
- *Ferromagnetic*: very large, non-linear magnetisation **M**.
  Permanent magnets are magnetised in the absence of external **B** field.

Diamagnetism (1)

In a varying magnetic field ($\vec{B} \parallel \vec{\mu}$), atomic electrons experience forces due to the induced electric field.



Faraday's law *(lecture 5)*:

$$\oint_L \vec{E} d\vec{l} = -\frac{d}{dt} \int_S \vec{B} d\vec{S}$$

E-field induced in the electron orbit along the direction of motion:

$$2\pi r E = \pi r^2 \frac{dB}{dt}$$, therefore $$E = \frac{r}{2} \cdot \frac{dB}{dt}$$ (up to the sign)

Equation of motion of the electron:

$$\underbrace{\tau = eEr}_{\text{Torque}} = \underbrace{m_e r^2}_{\text{Moment of inertia}} \frac{d\omega}{dt}$$, therefore $$\frac{er^2}{2} \frac{dB}{dt} = m_e r^2 \frac{d\omega}{dt}$$

Diamagnetism (2)

If the magnetic field varies from 0 to B over a time t,
the change of angular frequency of electron rotation is

$$\Delta\omega = \int_0^t \frac{d\omega}{dt}\cdot dt = \frac{e}{2m_e}\int_0^t \frac{dB}{dt}\cdot dt = \frac{eB}{2m_e}$$

This quantity is also known as the *Larmor angular frequency*:

$$\omega_L = \frac{eB}{2m_e} = -\Gamma B$$

Lenz rule: the induced magnetic moment $\vec{\mu}$ is antiparallel to $\vec{B}$.

Therefore, the induced angular moment $\vec{L}$ is parallel to $\vec{B}$.

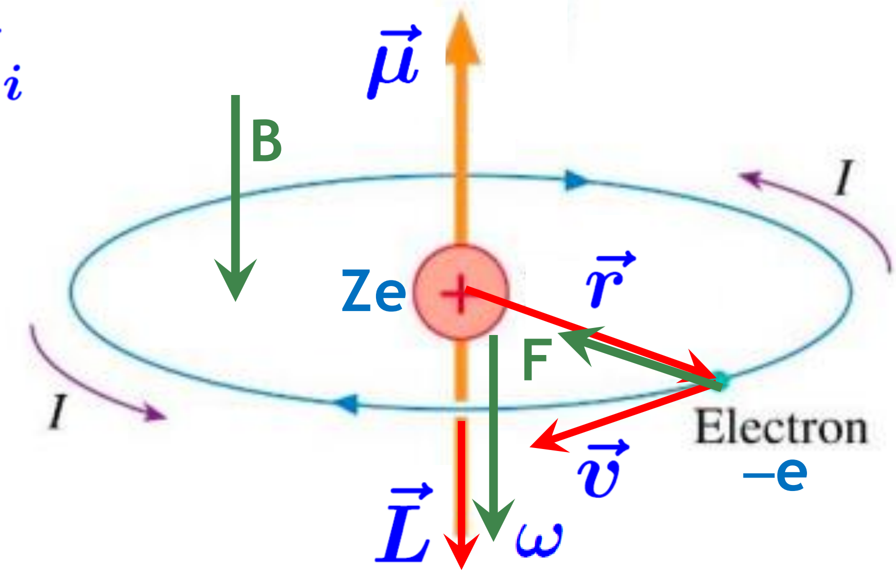
$$\vec{\mu} = \Gamma\vec{L} = -\frac{e}{2m_e}\cdot m_e\Delta\vec{\omega}r^2 = -\frac{e}{2m_e}\cdot m_e\frac{e\vec{B}}{2m_e}r^2 = -\frac{e^2r^2}{4m_e}\vec{B}$$

Diamagnetism ($\chi_B<0$) is a universal phenomenon,
and is a consequence of electromagnetic induction.
Diamagnetic materials are (weakly) repelled by magnetic fields.

Diamagnetism, via Lorentz force

[Not discussed in the lecture]

For each electron, $\vec{\mu}_i = -\dfrac{e}{2m_e}\vec{L}_i$



Diamagnetic materials:
electrons of atoms/molecules have zero total angular momentum, i.e. no permanent magnetic dipole moment

In the absence of external field, Newton's second law:

$$m_e a = m_e \omega_0^2 r = \frac{Ze^2}{4\pi\varepsilon_0 r^2} \; ; \qquad \omega_0 = \left(\frac{Ze^2}{4\pi\varepsilon_0 m_e r^3}\right)^{1/2}$$

Diamagnetism, via Lorentz force

[Not discussed in the lecture]

In external magnetic field, Lorentz force $\vec{F} = -e\vec{v} \times \vec{B}$

Newton's second law:

$$m_e \omega^2 r = \frac{Ze^2}{4\pi\varepsilon_0 r^2} + e\omega r B\,; \quad \omega^2 - \frac{eB}{m_e}\omega - \omega_0^2 = 0$$

Then $\omega^2 - 2\omega_L \omega - \omega_0^2 = 0$

The positive solution of this quadratic equation:

$$\omega = \omega_L + \sqrt{\omega_0^2 + \omega_L^2} \approx \omega_L + \omega_0 \left(1 + \frac{1}{2}(\omega_L/\omega_0)^2\right) =$$

$$\omega_0 + \omega_L \left(1 + \frac{\omega_L}{2\omega_0}\right) \approx \omega_0 + \omega_L$$

If $\vec{\omega}$ is in the direction of $\vec{B}$ it *increases* by ω_L.

If $\vec{\omega}$ is in the opposite direction, it *decreases* by ω_L.

Magnetisation of diamagnetics

Induced magnetic moment for a single electron: $\vec{\mu} = -\frac{e^2 r^2}{4m_e}\vec{B}$

For an atom, $\vec{\mu}_{\text{atom}} = -\frac{Ze^2 r_0^2}{6m_e}\vec{B}$

(Z: atomic number; r_0: root-mean-square radius of electron orbit;

1/6 comes from $\overline{x^2} + \overline{y^2} = \frac{2}{3} \cdot \overline{R^2}$, due to the spherical symmetry).

Magnetisation: induced magnetic dipole moment per unit volume

$$\vec{M} = -\frac{nZe^2 r_0^2}{6m_e}\vec{B}$$ (n: density of atoms [m^{-3}])

Magnetic susceptibility for diamagnetics: $\chi_B = -\frac{\mu_0 n Z e^2 r_0^2}{6m_e}$

- ❖ Depends on density only (not on temperature or pressure).
- ❖ Measurements of χ_B are used to determine atomic sizes.
  E.g. neon at STP: $\chi_B = -3.76\times10^{-9}$ leads to $r_0 = 0.49\times10^{-9}$ m.

Paramagnetism

Paramagnetic materials:

non-zero atomic permanent magnetic dipole moments (e.g. odd number of electrons; ions and ionic molecules).
Permanent magnetic moments tend to align with the field *(lecture 5)*

Typical atomic magnetic dipole moments are of the order μ_B
For a very strong field (**B=10 T**), work required to revert the direction of magnetic moment μ:

$$W \approx 2\mu_B B = 2 \cdot 9.27 \times 10^{-24}\ \mathrm{A \cdot m^2} \cdot 10\ \mathrm{T} \sim 10^{-3}\ \mathrm{eV}$$

Much smaller than thermal motion energy at room temperature:

$$\frac{3}{2}kT \approx 0.04\ \mathrm{eV}$$

Polarisation of polar dielectrics *(lecture 7)*: $\vec{P} = \frac{np^2}{3kT}\vec{E}$

Exact analogy for paramagnetic materials: $\vec{M} = \frac{n\mu^2}{3kT}\vec{B}$

Magnetisation of paramagnetics

Magnetic susceptibility for *paramagnetic materials*:

$$\chi_B = \mu_0 n \left(\frac{\mu^2}{3kT} - \frac{Ze^2 r_0^2}{6m_e} \right) > 0$$

Paramagnetic term
due to alignment of permanent
atomic magnetic moments (μ)

Diamagnetic term
due to induced
magnetic moments

- Materials with $\mu=0$ are usually diamagnetic.
- For most materials with $\mu \neq 0$, the paramagnetic term dominates over the diamagnetic term.
- This leads to the *Curie's law* ($\chi_B \sim 1/T$) for gaseous paramagnetics at $\mu B \ll kT$ (i.e. at large T).
- Paramagnetic materials are (weakly) attracted by magnetic fields.

Summary

- External magnetic fields induce magnetisation of materials, which is characterised by the *magnetic susceptibility* χ_B:

$$\vec{M} = \chi_B \frac{\vec{B}}{\mu_0}$$

- *Diamagnetism* ($\chi_B < 0$) is a universal phenomenon: *induced magnetic dipole moments* are directed against the **B** field.

- *Paramagnetism* ($\chi_B > 0$) is due to alignment of *permanent atomic magnetic moments* along the **B** field.

- Microscopic theory of magnetism leads to

$$\chi_B = \mu_0 n \left(\frac{\mu^2}{3kT} - \frac{Ze^2 r_0^2}{6m_e} \right)$$