UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 10
Magnetic field in materials (part 1)

- Magnetised materials in non-uniform fields
- Surface and volume magnetisation currents
- The auxiliary **H** field
- Ampere–Maxwell law for the **H** field

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Previous lecture

- External magnetic fields induce magnetisation of materials, which is characterised by the *magnetic susceptibility* χ_B:

$$\vec{M} = \chi_B \frac{\vec{B}}{\mu_0}$$

- *Diamagnetism* ($\chi_B<0$) is a universal phenomenon: *induced magnetic dipole moments* are directed against the **B** field.

- *Paramagnetism* ($\chi_B>0$) is due to alignment of *permanent atomic magnetic moments* along the **B** field.

- Microscopic theory of magnetism leads to

$$\chi_B = \mu_0 n \left(\frac{\mu^2}{3kT} - \frac{Ze^2 r_0^2}{6m_e} \right)$$

Magnetised materials in B-field

Material placed in a non-uniform magnetic field, $\vec{B}(z) = B(z) \cdot \vec{e}_z$

Magnetisation gives rise to a magnetic force acting on the sample in a non-uniform magnetic field:

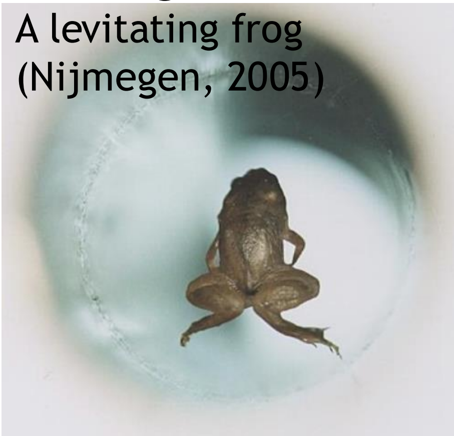
$$F_{\text{mag}} = \mu n V \cdot \frac{dB}{dz} = MV \cdot \frac{dB}{dz} = \frac{\chi_B B}{\mu_0} V \cdot \frac{dB}{dz}$$

μ: induced magnetic moment of an atom [A·m^2]
n: density of atoms [m^{-3}]
M=μn: magnetisation [A/m]
V: volume of the sample [m^3]

- Diamagnetics (χ_B<0) are repelled from regions of strong field.
- Paramagnetics (χ_B>0) are attracted into regions of strong field.

Levitation: magnetic force compensates gravity.

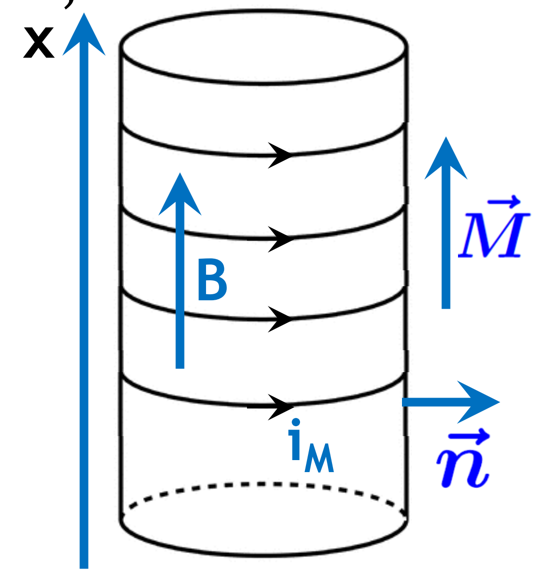
$$F_{\text{mag}} = mg = \rho V g \; ; \; B\frac{dB}{dz} = \frac{\mu_0 \rho g}{|\chi_B|}$$



For water ($\chi_B \approx -10^{-5}$), we need $B \cdot dB/dz \approx 1300$ T^2/m.
Much stronger effect for superconductors ($\chi_B = -1$).

Surface magnetisation currents

Consider a cylinder made of paramagnetic material, placed into a uniform magnetic field, parallel to the field lines.



Magnetic dipole moment of a horizontal slice δx:

$$m = \underbrace{\pi r^2 M \delta x}_{\text{By definition of } M,\ \text{magnetisation x volume}} = \underbrace{\pi r^2 i_M \delta x}_{\text{By definition of } m,\ \text{area x current}}$$

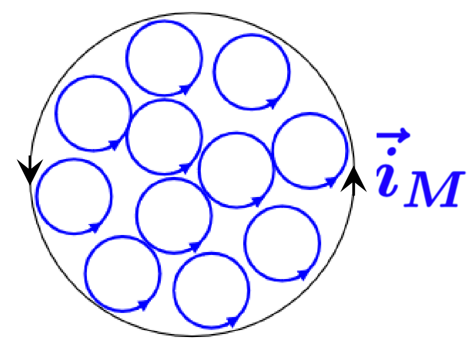
By definition of M, magnetisation x volume

By definition of m, area x current *(lecture 5)*

Surface current density: $i_M = M$ [A/m]

In the general case, $\boxed{\vec{i}_M = \vec{M} \times \vec{n}}$

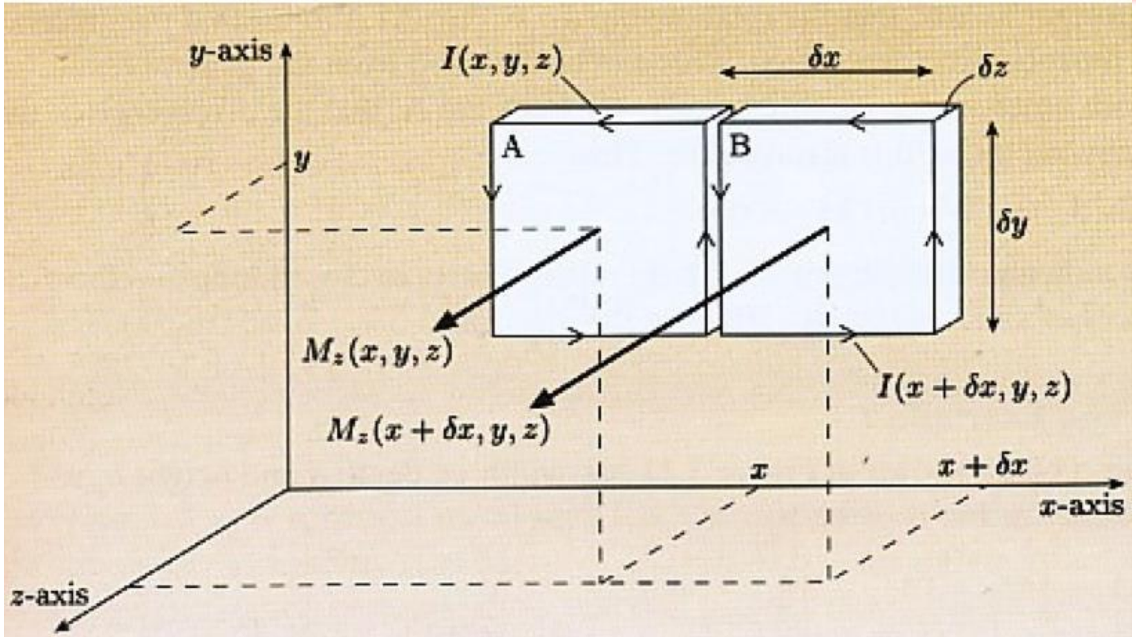
$\vec{n}$: unit normal vector pointing outwards



Top view

- Surface currents are perpendicular to M.
- Currents flow on any surface which is **not** perpendicular to M.

Volume magnetisation currents (1)



Volume currents flow in *non-uniformly magnetised* materials.

Assume magnetisation $\vec{M}$ along the **z** axis.

Consider two adjacent identical small volumes A, B.

Magnetic moments of the volumes:

$$m_A = M_z(x, y, z) \cdot dxdydz$$

$$m_B = M_z(x + dx, y, z) \cdot dxdydz = m_A + \left(\frac{\partial M_z}{\partial x} dx\right) \cdot dxdydz$$

Current in loop B:

$$I_B = \frac{m_B}{dxdy} = \frac{m_A + \frac{\partial M_z}{\partial x} dx \cdot dxdydz}{dxdy} = I_A + \frac{\partial M_z}{\partial x} dxdz$$

Volume magnetisation currents (2)

Net current in the y direction at the boundary between two volumes:

$$I_y = I_A - I_B = -\frac{\partial M_z}{\partial x} dx dz$$

Electric current density, by definition: $j_y = \frac{I_y}{dx dz} = -\frac{\partial M_z}{\partial x}$

There is similar contribution to j_y from the dependence of the magnetisation component M_x on z, therefore

$$j_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}$$

Similarly for the other components of j_M:

$$j_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \quad \text{and} \quad j_z = \frac{\partial M_y}{\partial z} - \frac{\partial M_z}{\partial y}$$

Finally, $\vec{j}_M = \nabla \times \vec{M}$

Analogy with the electric field

Magnetisation of the sample, **M(x,y,z)**, can be replaced by an equivalent distribution of *magnetisation currents*:

$$\vec{i}_M = \vec{M} \times \vec{n} \qquad \vec{j}_M = \nabla \times \vec{M}$$

This is similar to the electric field *(lecture 7)*:
polarisation of the sample, **P(x,y,z)**, can be replaced by an equivalent distribution of *polarisation charges*:

$$\sigma_{\mathrm{p}} = \vec{P} \cdot \vec{n} \qquad \rho_{\mathrm{p}} = -\nabla \vec{P}$$

Let's introduce an auxiliary magnetic field, **H**, to facilitate field calculations, similarly to the electric displacement field **D**.

The H-field; Ampere–Maxwell law

Maxwell's equation **M4** *(lecture 6)*; the current is the sum of conduction ("free"), polarisation and magnetisation currents:

$$\nabla \times \vec{B} = \mu_0(\vec{j}_\mathrm{f} + \vec{j}_\mathrm{p} + \vec{j}_M) + \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} =$$

$$\mu_0\vec{j}_\mathrm{f} + \mu_0\frac{\partial \vec{P}}{\partial t} + \mu_0\nabla \times \vec{M} + \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} = \mu_0\vec{j}_\mathrm{f} + \mu_0\frac{\partial \vec{D}}{\partial t} + \mu_0\nabla \times \vec{M}$$

See lecture 7

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

(lecture 8)

Therefore $\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{j}_\mathrm{f} + \frac{\partial \vec{D}}{\partial t}$

Let's define an auxiliary **H** field: $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

Then $\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$

which replaces **(M4)**, accounting for electric and magnetic properties of the medium (the index "**f**" is implied)

Displacement current density in matter *(see also lecture 6)*

Summary

- Volume distribution of magnetisation **M** can be replaced by an equivalent distribution of *magnetisation currents*:

$$\vec{i}_M = \vec{M} \times \vec{n} \qquad \vec{j}_M = \nabla \times \vec{M}$$

- The auxiliary **H**-field is defined as

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$, or equivalently $$\vec{B} = \mu_0(\vec{H} + \vec{M})$$

- In dielectric and magnetic media, Maxwell's equation **(M4)** becomes

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$