UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 11
Magnetic field in materials (part 2)

- Relative permeability of materials
- Boundary conditions for the **B** and **H** fields
- Energy density of the magnetic field

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Previous lecture

- Volume distribution of magnetisation **M** can be replaced by an equivalent distribution of *magnetisation currents*:

$$\vec{i}_M = \vec{M} \times \vec{n} \qquad \vec{j}_M = \nabla \times \vec{M}$$

- The auxiliary **H**-field is defined as

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$, or equivalently $$\vec{B} = \mu_0(\vec{H} + \vec{M})$$

- In dielectric and magnetic media, Maxwell's equation **(M4)** becomes

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

Physical meaning

Physical meaning:

- The B-field [T] (aka “magnetic flux density”): *force* acting on unit charge.
- Magnetisation field M [A/m]: *magnetic dipole moment* per unit volume, a local property of the material.
- The H-field (aka “magnetic field strength”) [A/m]: facilitates calculations in the presence of media (magnetisation & polarisation currents do not enter eq. M4). Magnetostatics: conduction current is the source of the H-field.

Absence of magnetic poles (eq. M2):

$$\nabla \vec{B} = \mu_0 \cdot \nabla(\vec{H} + \vec{M}) = 0$$

Therefore $\nabla \vec{H} = -\nabla \vec{M}$ (non-zero in general)

H and M field lines are not continuous: M changes abruptly at boundaries of media, and H compensates for this.

Magnetic susceptibility

Magnetic susceptibility, χ_B, was defined in *lecture 9* as

$$\vec{M} = \chi_B \frac{\vec{B}}{\mu_0}$$

Let's redefine the susceptibility as $\vec{M} = \chi_M \vec{H}$

The historic χ_M definition is widely used in literature, alongside χ_B.

For both dia- and paramagnetics, $|\chi_M| \ll 1, \ |\chi_B| \ll 1$.

Therefore, *the two definitions are equivalent* in practice.

The definition of the H-field, $\vec{B} = \mu_0(\vec{H} + \vec{M})$,

leads to $\dfrac{\mu_0}{\chi_B}\vec{M} = \mu_0\left(\dfrac{1}{\chi_M}\vec{M} + \vec{M}\right); \quad \dfrac{1}{\chi_B} = 1 + \dfrac{1}{\chi_M}$

Finally, $\chi_B = \dfrac{\chi_M}{1+\chi_M} \approx \chi_M$

Relative permeability

Using $\vec{M} = \chi_M \vec{H}$, we obtain

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1 + \chi_M)\vec{H} = \mu_0 \mu \vec{H}$$

where $\mu = 1 + \chi_M$ is the *relative permeability* of the material.

Ampere's law of magnetostatics for uniform μ:

$$\nabla \times \vec{H} = \vec{j}_\mathrm{f} \text{ , therefore } \frac{1}{\mu_0 \mu} \nabla \times \vec{B} = \vec{j}_\mathrm{f}$$

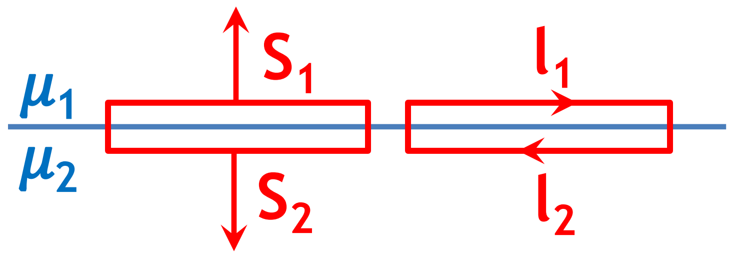
Finally, $\nabla \times \vec{B} = \mu_0 \mu \vec{j}_\mathrm{f}$

B-field *increases* by a factor of μ with respect to the field produced by the same distribution of free currents in free space.

This is contrary to the **E**-field, which *decreases* by a factor of ε.

Boundary conditions for B and H

Boundary of two media
with relative permeabilities μ_1 and μ_2



Absence of magnetic poles: the **B** field is solenoidal also in media.
Considering for a very thin cylinder,

$$\int_S \vec{B} d\vec{S} = 0 \quad \text{therefore} \quad \vec{B}_1\vec{S}_1 + \vec{B}_2\vec{S}_2 = 0$$

Therefore $\boxed{B_{1n} = B_{2n}}$

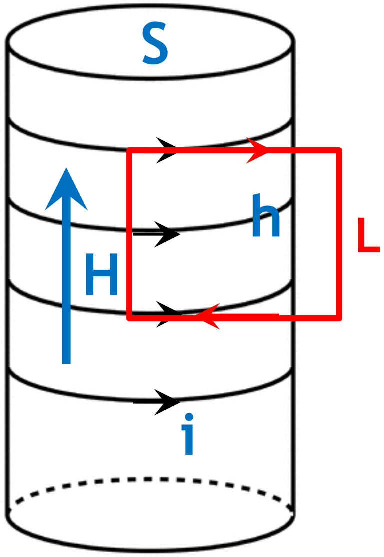
Ampere–Maxwell law for a very thin rectangular loop
(such that the **D**-field flux is negligibly small),
in the absence of surface conduction current:

$$\oint_L \vec{H} d\vec{l} = \vec{H}_1\vec{l}_1 + \vec{H}_2\vec{l}_2 = 0 \quad \text{similarly leads to} \quad \boxed{H_{1t} = H_{2t}}$$

Magnetic field energy (1)

Consider a current loop of area S, with negligible resistance.
To change the current in the loop (and the B-field),
work must be done by a current source against an induced emf V_{ind}:

$$\delta W = \underbrace{-IV_{\mathrm{ind}}dt = I\frac{d\Phi}{dt}dt}_{\text{Faraday's law (lecture 5)}} = Id\Phi = IS\cdot dB$$



Ampere's law for H field for a section of a long solenoid:

$$\oint_L \vec{H}d\vec{l} = \int_S \vec{j}_f d\vec{S}$$, leading to $Hh = I$

(I : total surface current in the section of height h)

Therefore $\delta W = Hh\cdot SdB = VHdB$ (V : volume)

Increment of *energy density* [J/m³] is $\delta w = HdB = \vec{H}d\vec{B}$

This is valid for dia- and paramagnetic media.
In ferromagnetic media, some energy is dissipated as heat.

Magnetic field energy (2)

For dia- and paramagnetics, $\vec{B} = \mu_0 \mu \vec{H}$ and the *energy density* is

$$w = \int_0^B dw = \int_0^B H dB = \mu_0 \mu \int_0^H H dH =$$

$$= \frac{1}{2} \mu_0 \mu H^2 = \frac{1}{2\mu_0 \mu} B^2 = \frac{1}{2} BH$$

A large energy in some cases.
Example: LHC superconducting magnets (B=8.3 T),
$w = B^2/2\mu_0 = (69 / (2\times1.26\times10^{-6}))$ J/m^3 = 27 MJ/m^3.

In magnetostatics, the energy density w can be interpreted as the potential energy of electric currents (assuming action at a distance).

Electrodynamic interpretation: *energy is localised in space* and is carried by the magnetic field.

Field energy in a solenoid

Definition of inductance: [Not discussed in the lecture]

$$L = \frac{\mathcal{E}}{dI/dt} = \frac{d\Phi_B/dt}{dI/dt} = \frac{d\Phi_B}{dI} = \frac{\Phi_B}{I}$$

For a long solenoid (length l; area S; N turns of wire with current I),

$$H = IN/l; \qquad B = \mu_0 \mu IN/l$$

Therefore,

$$L = \frac{\Phi_B}{I} = \frac{BNS}{I} = \frac{\mu_0 \mu I N^2 S}{Il} = \frac{\mu_0 \mu S N^2}{l}$$

The B-field, and therefore the inductance L, increase by a factor μ.

One can see that the total magnetic field energy is

(a non-assessed problem)

$$\underbrace{W = \frac{LI^2}{2}}_{\text{Standard formula from EM1}} = w \cdot Sl = \underbrace{\frac{BH}{2} \cdot Sl}_{\text{Energy density} \times \text{volume}}$$

Summary

- For dia- and paramagnetic materials, *magnetic susceptibility* χ_M and *relative permeability* μ are defined as

$$\vec{B} = \mu_0(1+\chi_M)\vec{H} = \mu_0\mu\vec{H}$$

- Boundary conditions for the **B** and **H** fields:

$$B_{1n} = B_{2n} \quad \text{and} \quad H_{1t} = H_{2t}$$

- Energy density of the magnetic field in dia- and paramagnetics:

$$w = \frac{1}{2}BH$$