UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 13
Electromagnetic waves in free space

- ❖ Wave equations in free space
- ❖ General solution to the wave equations
- ❖ Monochromatic plane wave solutions

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Previously: Maxwell's equations

The laws of electrodynamics (in matter):

(M1) $$\nabla \vec{D} = \rho$$

(M2) $$\nabla \vec{B} = 0$$

(M3) $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(M4) $$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

Boundary conditions:

$$D_{1n} = D_{2n}$$

$$E_{1t} = E_{2t}$$

$$B_{1n} = B_{2n}$$

$$H_{1t} = H_{2t}$$

For linear, isotropic and homogeneous materials, and weak fields varying slowly in time,

Ohm's law

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E}, \quad \vec{B} = \mu_0 \mu \vec{H}, \quad \vec{j} = \sigma \vec{E}$$

Waves in free space (1)

Assume no free charges ($\rho=0$), no free currents ($j=0$), and $\varepsilon=\mu=1$.

(M1) $$\nabla\vec{E}=0$$

(M3) $$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

(M2) $$\nabla\vec{B}=0$$

(M4) $$\nabla\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

Simplify the problem: assume $\vec{E}=\vec{E}(z,t);\quad \vec{B}=\vec{B}(z,t)$

Then for both E and B fields (denoted A below),

$$\nabla\vec{A}=\frac{\partial A_z}{\partial z}\,;\ \nabla\times\vec{A}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ 0 & 0 & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=-\vec{e}_x\frac{\partial A_y}{\partial z}+\vec{e}_y\frac{\partial A_x}{\partial z}$$

(no z component!)

Therefore (M1)–(M4) for the z-components of the fields become:

$$\frac{\partial E_z}{\partial z}=\frac{\partial B_z}{\partial z}=\frac{\partial E_z}{\partial t}=\frac{\partial B_z}{\partial t}=0$$

Uniform static E_z and B_z fields, not of any interest

Waves in free space (2)

y-component of (M3): $$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \quad (1)$$

x-component of (M4): $$-\frac{\partial B_y}{\partial z} = \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} \quad (2)$$

x-component of (M3): $$-\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad (3)$$

y-component of (M4): $$\frac{\partial B_x}{\partial z} = \varepsilon_0 \mu_0 \frac{\partial E_y}{\partial t} \quad (4)$$

Two independent pairs of equations: for (E_x, B_y) and for (E_y, B_x).

Exclude B_y from (1) and (2): $$-\frac{\partial^2 B_y}{\partial z \partial t} = \boxed{\frac{\partial^2 E_x}{\partial z^2} = \varepsilon_0 \mu_0 \frac{\partial^2 E_x}{\partial t^2}}$$

This is the *wave equation*. It is also valid for E_y, B_x, and B_y.

Wave equations: discussion

- The displacement current term in (M4), introduced by Maxwell, is essential for the propagation of waves.
- An accelerating electric charge is sufficient to generate an EM wave, which then propagates in space.
- Principle of superposition: a linear combination of two solutions $\{E_1(x,y,z,t), B_1(x,y,z,t)\}$ and $\{E_2(x,y,z,t), B_2(x,y,z,t)\}$ is also a solution to Maxwell's equations.
- Maxwell's equations are valid in any inertial reference frame. Therefore, EM waves propagate in free space at a fixed speed c, in any inertial reference frame (which is also postulated by special relativity).

Solutions to the wave equation

Let's introduce a quantity $c = 1/\sqrt{\varepsilon_0 \mu_0}$ [m/s]

The one-dimensional wave equation becomes $\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$

General (d'Alembert) solution: $E_x(z,t) = f_1(z-ct) + f_2(z+ct)$

Here $f_1(x)$ and $f_2(x)$ are smooth functions *(see the Maths 2B module)*

- Interpretation: an *electromagnetic wave* of a fixed smooth shape moving in the direction of **z** (for f_1) or **−z** (for f_2).
- These waves are of *transverse nature* (**E** and **B** are perpendicular to the direction of propagation, **z**).
- They propagate (in free space) at a *finite speed* of $c \approx 3\times10^8$ m/s.
- This speed agrees with the measured *speed of light in vacuum*. This led Maxwell to the hypothesis that **light is electromagnetic in nature** (**1865**).
- First experimental observation of EM waves: Hertz (**1886**).

The three-dimensional case

Assume no free charges ($\rho=0$), no free currents ($j=0$), and $\varepsilon=\mu=1$.

(M1) $\nabla\vec{E}=0$ (M3) $\nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$

(M2) $\nabla\vec{B}=0$ (M4) $\nabla\times\vec{B}=\varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t}$

To derive the equations for **B** and **E** fields separately, take curl of (M4)

$$\nabla\times(\nabla\times\vec{B})=\varepsilon_0\mu_0\nabla\times\frac{\partial\vec{E}}{\partial t}=\varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)=-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}$$

On the other hand, $\underbrace{\nabla\times(\nabla\times\vec{B})=\nabla(\nabla\vec{B})-\nabla^2\vec{B}}_{\text{See lecture 2}}=-\nabla^2\vec{B}$

See lecture 2

Therefore, $\nabla^2\vec{B}=\varepsilon_0\mu_0\dfrac{\partial^2\vec{B}}{\partial t^2}$ and similarly $\nabla^2\vec{E}=\varepsilon_0\mu_0\dfrac{\partial^2\vec{E}}{\partial t^2}$

Monochromatic plane waves

Plane wave solutions (*propagating in **z** direction*, without loss of generality):

$$\vec{E}(z,t) = \vec{E}_0 \cos(\omega t - kz) \qquad \boxed{\vec{E}_0 \perp \vec{e}_z}$$

Using *complex notation*, implying *the real part* of the expression:

$$\vec{E}(z,t) = \vec{E}_0 e^{i(\omega t - kz)}$$ (can be used for linear operations)

$$\frac{\partial^2 \vec{E}}{\partial z^2} = -k^2 \vec{E} \qquad \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E} \qquad \frac{\partial^2 \vec{E}}{\partial z^2} = \left(\frac{k}{\omega}\right)^2 \frac{\partial^2 \vec{E}}{\partial t^2}$$

A solution to the wave equation in case $\boxed{c = \omega / k}$

This is an important special case:

- ❖ waves far from the source of radiation are almost *plane waves*;
- ❖ a wave packet can be decomposed into *monochromatic waves*.

$|\vec{E}_0|$: *amplitude* of the electric field [V/m]

$\omega = \frac{2\pi}{T}$: *angular frequency* [Hz = s^{-1}]; T is the period of oscillation. *Time-to-phase conversion.*

$k = \frac{2\pi}{\lambda}$: *wave number* [m^{-1}], i.e. the "spatial frequency"; λ is the wavelength. *Coordinate-to-phase conversion.*

Wave vector and wave fronts

Monochromatic plane wave *propagating in an arbitrary direction*:

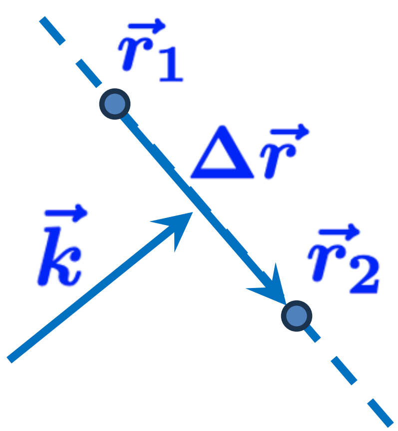
$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\omega t - \vec{k}\vec{r})}$$

Here $\vec{k}$ is the *wave vector*: it points in the direction of propagation, and its magnitude is equal to the wave number, $k = \omega / c$.

Wave front: a set of points where the wave has the same phase, ($\varphi=\omega t-kr$) at any fixed moment of time.

Wave front equation: $\vec{k}\vec{r} = \mathrm{const}$

This condition defines a plane perpendicular to $\vec{k}$.



Proof: for any two points (r_1,r_2) on a wave front,

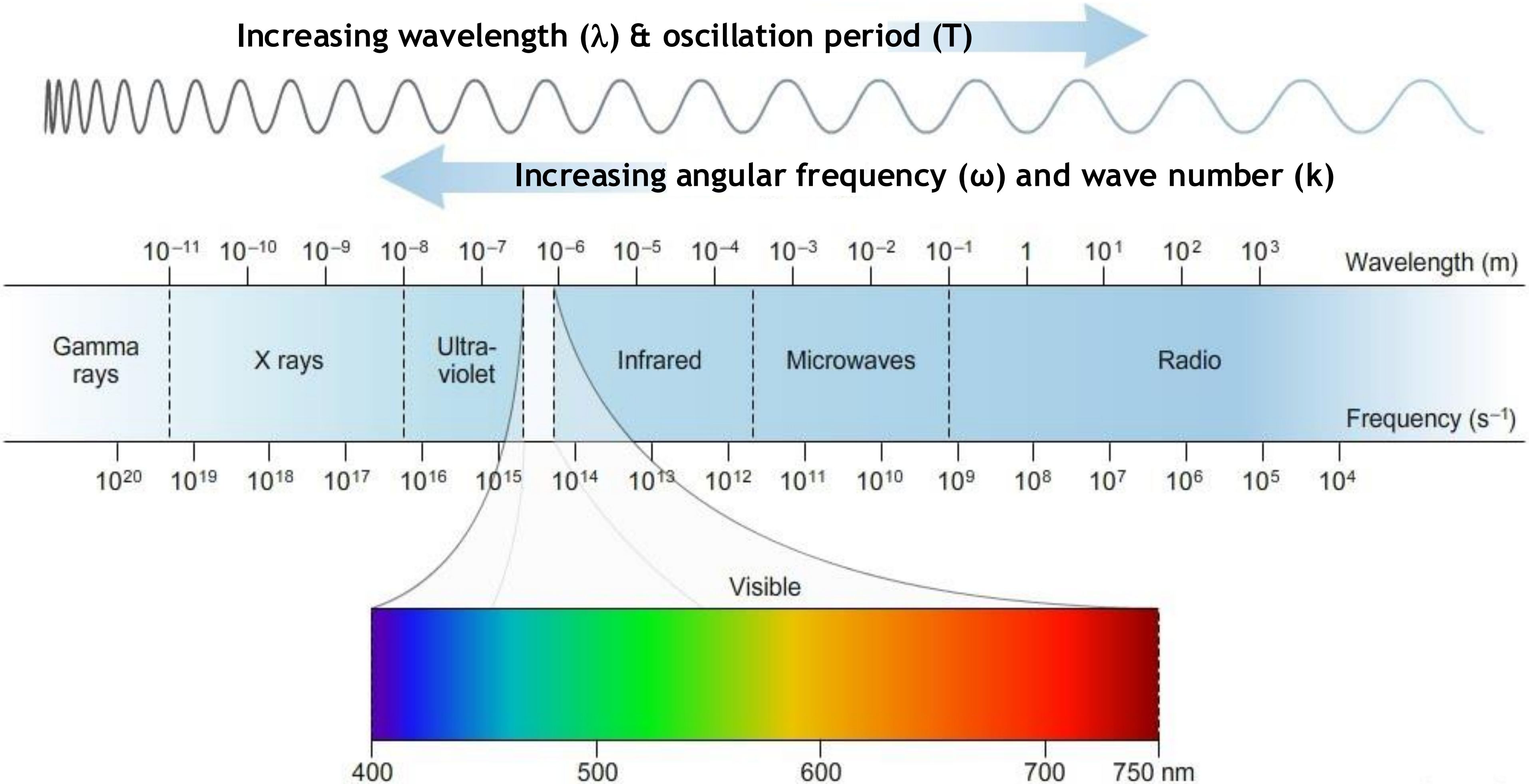
$$\vec{k}\cdot\vec{r}_1 = \vec{k}\cdot\vec{r}_2$$

Therefore, $\vec{k}\cdot(\vec{r}_2 - \vec{r}_1) = \vec{k}\cdot\Delta\vec{r} = 0$

This means $\vec{k} \perp \Delta\vec{r}$

The EM spectrum

No restrictions on the angular frequency $\omega = kc$:
the *electromagnetic spectrum*



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Summary

- *Wave equations* following from Maxwell's equations in free space:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

- Solutions: *EM waves* of transverse nature propagating with a *finite speed*, namely the speed of light

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

- Monochromatic plane wave solutions:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\omega t - \vec{k}\vec{r})}$$

with a *wave number* $k = \frac{\omega}{c}$, and a *wavelength* $\lambda = \frac{2\pi}{k}$.