UNIVERSITY OF
BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 14

EM waves in dielectrics; Poynting vector

- Properties of monochromatic plane waves in vacuum and dielectric materials
- Wave impedance
- EM energy flux; the Poynting vector

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Previous lecture

❖ *Wave equations* following from Maxwell's equations in free space:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

❖ Solutions: *EM waves* of transverse nature propagating with a *finite speed*, namely the speed of light

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

❖ Monochromatic plane wave solutions:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\omega t - \vec{k}\vec{r})}$$

with a *wave number* $k = \frac{\omega}{c}$, and a *wavelength* $\lambda = \frac{2\pi}{k}$.

Plane waves in vacuum (1)

A plane monochromatic wave propagating in the direction $\vec{k}$:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\omega t - \vec{k}\vec{r})}$$

Let's write down the individual components:

$$E_x = E_{0x} e^{i(\omega t - k_x x - k_y y - k_z z)}$$

$$E_y = E_{0y} e^{i(\omega t - k_x x - k_y y - k_z z)}$$

$$E_z = E_{0z} e^{i(\omega t - k_x x - k_y y - k_z z)}$$

Let's find the expression for $\nabla \vec{E}$:

$$\frac{\partial E_x}{\partial x} = -ik_x E_{0x} e^{i(\omega t - k_x x - k_y y - k_z z)} = -ik_x E_x$$

and similarly for the other derivatives. Therefore

$$\nabla \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -ik_x E_x - ik_y E_y - ik_z E_z = -i\vec{k}\vec{E}$$

Plane waves in vacuum (2)

Identities for **E** and **B** fields in plane monochromatic waves:

(Prove them: a non-assessed problem)

$$\nabla \vec{E} = -i\vec{k}\vec{E}$$

$$\nabla \times \vec{E} = -i\vec{k} \times \vec{E}$$

$$\nabla^2 \vec{E} = -k^2 \vec{E}$$

$$\frac{\partial \vec{E}}{\partial t} = i\omega \vec{E}$$

Shortcuts:

$$\nabla \longrightarrow -i\vec{k}$$

$$\frac{\partial}{\partial t} \longrightarrow i\omega$$

(M1) in absence of charges: $\nabla \vec{E} = 0$

Therefore $\vec{k}\vec{E} = 0$.

Similarly, (M2) leads to $\vec{k}\vec{B} = 0$.

This proves the transverse nature of plane EM waves: $\vec{E}, \vec{B} \perp \vec{k}$

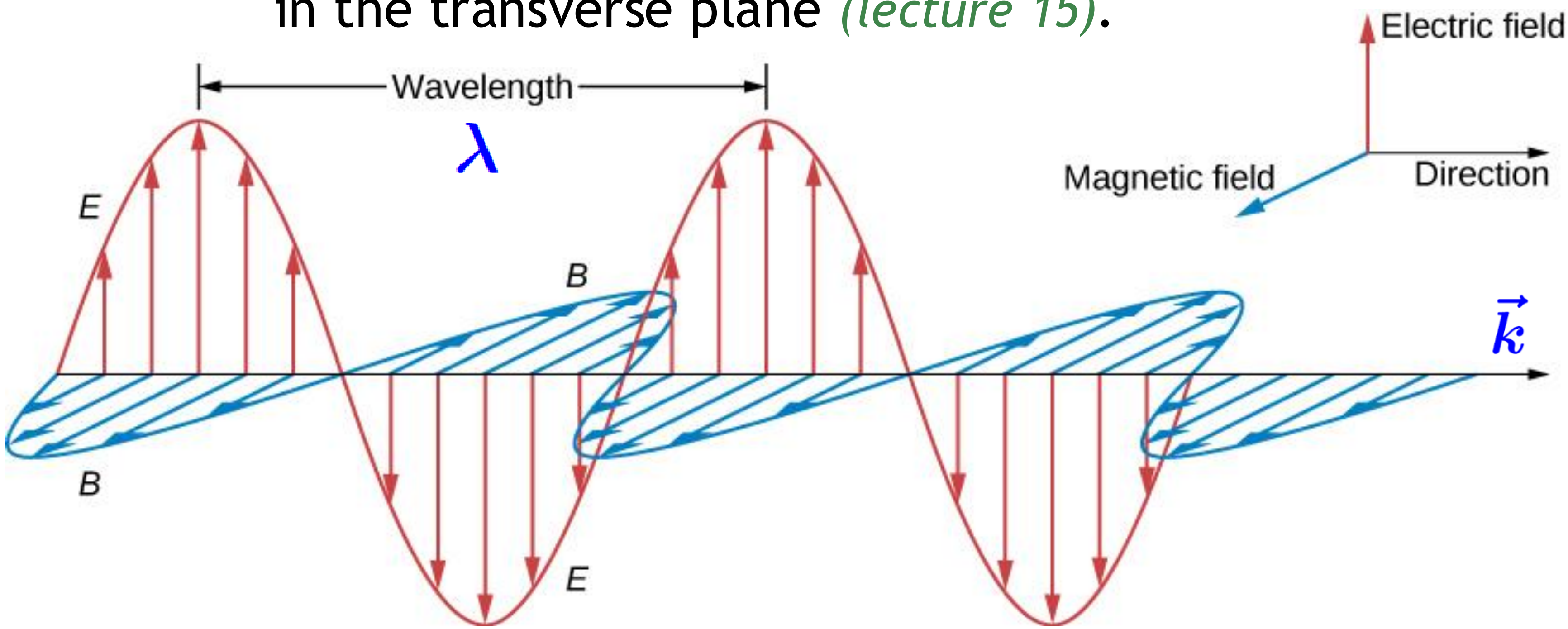
Equation (M3), $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

leads to $-i\vec{k} \times \vec{E} = -i\omega \vec{B}$

Therefore $\vec{B} = \dfrac{1}{\omega}\left(\vec{k} \times \vec{E}\right)$; equivalently, $c\vec{B} = \left(\hat{\vec{k}} \times \vec{E}\right)$.

Plane waves in vacuum (3)

- The ratio of E to B fields is always $E(\vec{r},t)/B(\vec{r},t) = c$.
- Therefore, the E and B fields *oscillate in phase*.
- The (E, B, k) vectors are mutually orthogonal, and form a right-handed system.
- Summary of these properties: $\vec{B} = \frac{1}{\omega}\left(\vec{k} \times \vec{E}\right)$ or $c\vec{B} = \left(\hat{\vec{k}} \times \vec{E}\right)$.
- However, the E and B vectors can rotate simultaneously in the transverse plane *(lecture 15)*.



Wave equation in dielectrics

Consider a *LIH dielectric*, with no free charges and no free currents.
Difference to free space: **ε≠1**, ***μ*≠1**.

Considering that $\vec{D} = \varepsilon_0 \varepsilon \vec{E}, \quad \vec{B} = \mu_0 \mu \vec{H},$

Eq. **(M4)** $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$ becomes $\nabla \times \vec{B} = \varepsilon \varepsilon_0 \mu \mu_0 \frac{\partial \vec{E}}{\partial t}$

The wave equation is obtained analogously to *lecture 13*:

$$\nabla^2 \vec{E} = \varepsilon \varepsilon_0 \mu \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \varepsilon \varepsilon_0 \mu \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

EM waves propagate with a speed $v = \frac{1}{\sqrt{\varepsilon \varepsilon_0 \mu \mu_0}} = \frac{c}{\sqrt{\varepsilon \mu}}$

Plane wave solutions: similar to those in free space,

$$\vec{E}(z,t) = \vec{E}_0 e^{i(\omega t - kz)}$$

with a wave number $k = \frac{\omega}{v} = \sqrt{\varepsilon \mu} \cdot \frac{\omega}{c}$

Plane waves in dielectrics

	Vacuum ($\varepsilon=\mu=1$)	Dielectrics ($\varepsilon\neq1$, $\mu\neq1$)
Speed; E/B ratio	c	$v = c/\sqrt{\varepsilon\mu} = c/n$
Wave number	$k = \frac{\omega}{c}$	$k = \frac{\omega}{v} = \frac{\omega}{c}\sqrt{\varepsilon\mu} = \frac{\omega n}{c}$
Wavelength	$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega}$	$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega}\frac{1}{\varepsilon\mu} = \frac{2\pi c}{n\omega}$

- The *refractive index* of the medium: $n = \sqrt{\varepsilon\mu} \approx \sqrt{\varepsilon}$
- For a fixed angular frequency ω, wave speed v and wavelength λ in a dielectric medium are *smaller* than in vacuum by a factor n.
- Boundaries of media: the same angular frequency ω on both sides. Therefore, different wave number (k) and wavelength (λ).
- As in free space, E and B oscillate in phase, (E, B, k) is a RH system and $\vec{B} = \frac{1}{\omega}\left(\vec{k} \times \vec{E}\right)$; however $v\vec{B} = \left(\vec{\hat{k}} \times \vec{E}\right)$.

Wave impedance

Wave impedance is the ratio of field magnitudes: $Z = E/H$

[(V/m)/(A/m) = Ω]

- For waves in free space, the *impedance of free space*:

$$Z_0 = \frac{E}{H} = \frac{\mu_0 E}{B} = \mu_0 c = \mu_0/\sqrt{\varepsilon_0\mu_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega$$

- For waves in dielectrics, $Z = \frac{E}{H} = \frac{E}{B}\cdot\frac{B}{H} = v\cdot\mu\mu_0 =$

$$\frac{1}{\sqrt{\varepsilon\varepsilon_0\mu\mu_0}}\cdot\mu\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot\sqrt{\frac{\mu}{\varepsilon}} = Z_0\sqrt{\frac{\mu}{\varepsilon}} \approx \frac{Z_0}{n}$$

Ratio of electric to magnetic energy densities:

$$R = \frac{ED}{BH} = \frac{\varepsilon\varepsilon_0 E^2}{\mu\mu_0 H^2} = Z^2\frac{\varepsilon\varepsilon_0}{\mu\mu_0} = \frac{\mu\mu_0}{\varepsilon\varepsilon_0}\cdot\frac{\varepsilon\varepsilon_0}{\mu\mu_0} = 1$$

(exactly)

- In conducting materials, **E** and **H** are in general out of phase, and the impedance **Z=E/H** is a complex number.

Energy flow in EM field

Elementary change of energy density (valid for any medium):

$$du = \underbrace{\vec{E}d\vec{D}}_{\text{Lecture 8}} + \underbrace{\vec{H}d\vec{B}}_{\text{Lecture 11}} + \underbrace{q}_{\text{Joule heat per unit volume}}$$

Joule heat in a small volume dV over a small period of time dt:

$$dQ = qdV = \underbrace{\rho dV}_{\text{Electric charge}} \cdot \vec{E} \cdot \vec{v}dt = \vec{j}\vec{E} \cdot dV\,dt$$

Therefore $q = \vec{j}\vec{E} \cdot dt$

Rate of change of the *total* energy per unit volume (EM + heat):

$$\frac{\partial u}{\partial t} = \vec{E}\frac{\partial \vec{D}}{\partial t} + \vec{H}\frac{\partial \vec{B}}{\partial t} + \vec{j}\vec{E} = \vec{E}\left(\frac{\partial \vec{D}}{\partial t} + \vec{j}\right) + \vec{H}\frac{\partial \vec{B}}{\partial t}$$

$$= \underbrace{\vec{E}(\nabla \times \vec{H})}_{\text{(M4)}} - \vec{H}\underbrace{(\nabla \times \vec{E})}_{\text{(M3)}} = \underbrace{-\nabla(\vec{E} \times \vec{H})}_{\text{The cross-product rule}}$$

The cross-product rule:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

The Poynting vector

Let's define the *Poynting vector*: $\vec{N} = \vec{E} \times \vec{H}$ [V/m × A/m = W/m^2]

Then $\nabla \vec{N} = -\frac{\partial u}{\partial t}$: this is the continuity equation *(lecture 6)*

Cf. the law of conservation of electric charge: $\nabla \vec{j} = -\frac{\partial \rho}{\partial t}$

Therefore, $\vec{N}$ is interpreted as the *energy flux*, i.e.
the rate of energy transfer per unit area per unit time.

For example, the solar constant is **1.36 kW/m^2**.

The continuity equation $\nabla \vec{N} = -\frac{\partial u}{\partial t}$ holds if $\vec{N}$ is replaced with

$$\vec{N}' = \vec{N} + \nabla \times \vec{A}$$, therefore $\nabla \vec{N}' = \nabla \vec{N}$

(divergence of curl is zero, lecture 2)

The definition of the Poynting vector is not unique:
this difficulty is resolved in the relativistic EM field theory.

Summary

- Useful shortcuts for plane monochromatic plane waves:

$$\nabla \longrightarrow -i\vec{k} \qquad \frac{\partial}{\partial t} \longrightarrow i\omega$$

- Relation between the **E** and **B** fields for plane monochromatic waves in vacuum and dielectrics:

$$\vec{B} = \frac{1}{\omega}\left(\vec{k} \times \vec{E}\right) \quad \text{or} \quad v\vec{B} = \left(\vec{\hat{k}} \times \vec{E}\right)$$

(**E**, **B**, **k**) are mutually perpendicular, form a right-hand system, oscillate in phase, and **E/B=ω/k=v**.

- At fixed **ω**, wave speed and wavelength are reduced in dielectrics wrt vacuum by a factor $n = \sqrt{\varepsilon\mu}$ (the refractive index):

$$v = c/\sqrt{\varepsilon\mu} = c/n$$

- The EM energy flux is quantified by the *Poynting vector*:

$$\vec{N} = \vec{E} \times \vec{H}$$