UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 15
Polarisation of plane waves

- Energy flux in a monochromatic plane wave
- Linear, circular and elliptic polarisations
- Absorptive linear polarisers
- Circular polarisers: quarter-wave plates

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Previous lecture

- Useful shortcuts for plane monochromatic plane waves:

$$\nabla \longrightarrow -i\vec{k} \qquad \frac{\partial}{\partial t} \longrightarrow i\omega$$

- Relation between the **E** and **B** fields for plane monochromatic waves in vacuum and dielectrics:

$$\vec{B} = \frac{1}{\omega}\left(\vec{k} \times \vec{E}\right) \quad \text{or} \quad v\vec{B} = \left(\vec{\hat{k}} \times \vec{E}\right)$$

(**E**, **B**, **k**) are mutually perpendicular, form a right-hand system, oscillate in phase, and **E/B=ω/k=v**.

- At fixed **ω**, wave speed and wavelength are reduced in dielectrics wrt vacuum by a factor $n = \sqrt{\varepsilon\mu}$ (the refractive index):

$$v = c/\sqrt{\varepsilon\mu} = c/n$$

- The EM energy flux is quantified by the *Poynting vector*:

$$\vec{N} = \vec{E} \times \vec{H}$$

Energy flux in a plane wave

For a monochromatic plane wave,

$$\vec{N} = \vec{E} \times \vec{H} = \frac{1}{\mu_0\mu}\vec{E} \times \vec{B} = \frac{1}{\mu_0\mu v}\vec{E} \times \left(\vec{\hat{k}} \times \vec{E}\right)$$

$$= \frac{E^2}{\mu_0\mu v}\vec{\hat{k}} = \frac{E^2}{\mu_0\mu v^2}\vec{v} = \frac{E^2}{\mu_0\mu} \cdot \varepsilon_0\varepsilon\mu_0\mu \cdot \vec{v} = \varepsilon_0\varepsilon E^2\vec{v}$$

Here $\vec{\hat{k}}$ is a unit vector in the direction of propagation.

The energy density is $\underbrace{u = 2u_{\mathrm{E}}}_{\text{Lecture 14}} = ED = \varepsilon_0\varepsilon E^2$

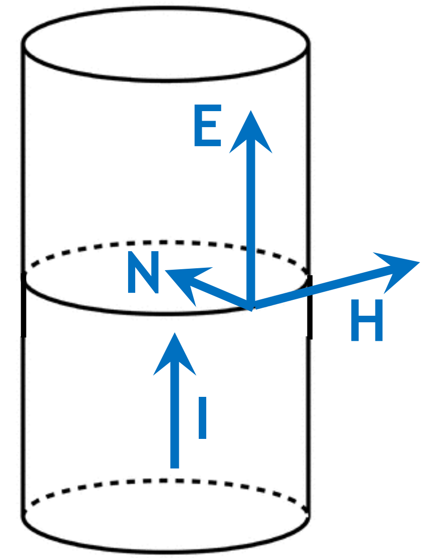
Therefore $\boxed{\vec{N} = u\vec{v}}$ [W/m^2 = J/m^3 × m/s]

The **E**-field oscillates, and the energy flux (**N~E^2**) depends on time. At a fixed point in space, averaged over an oscillation period,

$$\langle\vec{N}\rangle = \varepsilon_0\varepsilon\vec{v}\langle E^2\rangle = \varepsilon_0\varepsilon\vec{v}\langle E_0^2\cos^2(\omega t)\rangle = \frac{1}{2}E_0^2\varepsilon_0\varepsilon\vec{v}$$

Poynting vector: further examples

1) Steady current flowing in a cylindrical wire of radius **R** and length **L**: the **N** vector points inwards.



From Ampere's law, $2\pi RH = I$ and $H = \dfrac{I}{2\pi R}$

Side area of the wire: $S = 2\pi RL$

Energy flow into the wire per unit time:

$$W = NS = EHS = E \cdot \frac{I}{2\pi R} \cdot 2\pi RL = IEL = IU,$$

where **U=EL** is the voltage drop along the wire.
EM energy flows into the wire from outside and becomes Joule heat.

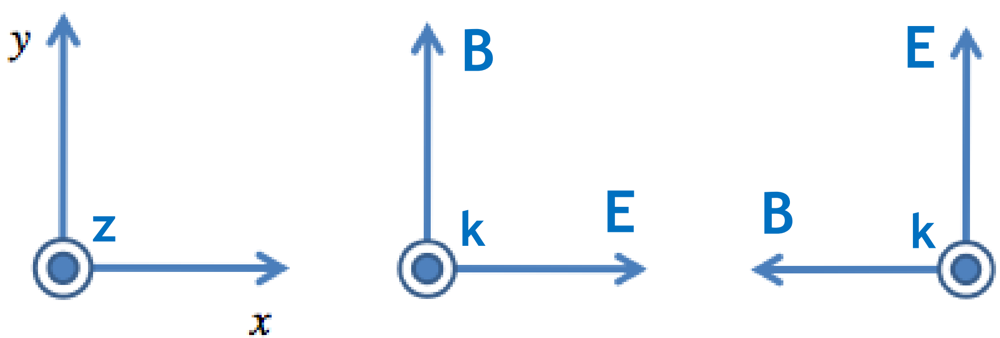
2) A charged cylindrical capacitor in a magnetic field **H** parallel to its axis: the energy flows in circles.
Integrated over the volume, zero energy flux and zero momentum, but **non-zero angular momentum** of the EM field.

Polarisation of plane waves

Consider a plane wave propagating in the +z direction.
There are two independent *linearly polarised* waves:

$$\vec{E}_x = \vec{e}_x E_{x0} e^{i(wt-kz)}$$, E vector always in the x direction

$$\vec{E}_y = \vec{e}_y E_{y0} e^{i(wt-kz)}$$, E vector always in the y direction



$$v\vec{B} = \left(\vec{\hat{k}} \times \vec{E}\right)$$

i.e. (E,B,k) form a right-handed system (z,k directed towards us)

General description of a plane wave propagating the +z direction:

$$\vec{E}(z,t) = (\vec{e}_x E_{x0} + \vec{e}_y E_{y0}) e^{i(\omega t - kz)}$$

In the general case, E_{x0} and E_{y0} are *complex amplitudes*,

$$E_{x0} = |E_{x0}| e^{i\varphi_{x0}}, \quad E_{y0} = |E_{y0}| e^{i\varphi_{y0}}$$

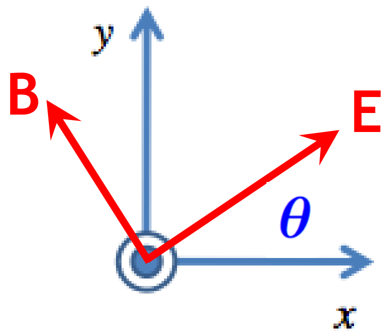
allowing for a phase difference between the two waves.

Types of polarisation

1) *Linearly polarised* waves.

E_{x0} and E_{y0} have the same phase ($\varphi_{x0}=\varphi_{y0}$).

The E_0 vector is at an angle $\theta = \arctan(E_{y0}/E_{x0})$ to the x axis, and has a magnitude of $\sqrt{E_{x0}^2 + E_{y0}^2}$



2) *Circularly polarised* waves.

$|E_{x0}| = |E_{y0}|$; phase difference $|\varphi_{x0} - \varphi_{y0}| = \dfrac{\pi}{2}$

Then, up to a constant phase,

$$\vec{E}(z,t) = |E_{x0}|(\vec{e}_x \pm i\vec{e}_y)e^{i(\omega t - kz)}$$

The field magnitude is the real part of this expression *(lecture 13)*:

$$E_x(z,t) = |E_{x0}|\cos(\omega t - kz)$$

$$E_y(z,t) = |E_{x0}|\cos\left(\omega t - kz \pm \frac{\pi}{2}\right) = \mp|E_{x0}|\sin\left(\omega t - kz\right)$$

3) *Elliptically polarised* waves: any other case.

Circular polarisation

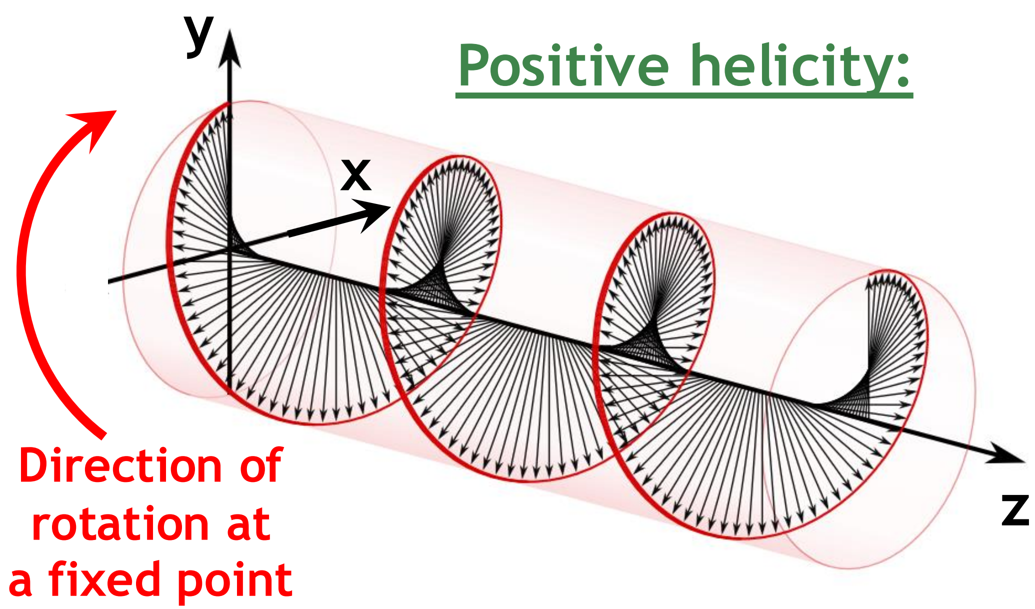
For circular polarisation, at a fixed transverse plane z=0,

$$E_x(t) = E_0 \cos(\omega t), \quad E_y(t) = \mp E_0 \sin(\omega t)$$

The field vector has a fixed magnitude E_0,
and sweeps around a circle with an angular frequency ω.
Energy flux ($N \sim E^2$) is constant in time.

Upper sign (–):
"left circularly polarised"
or "positive helicity"
[counter-clockwise when facing the oncoming wave]

Lower sign (+):
"right circularly polarised"
or "negative helicity"



Elliptical polarisation: the field vector sweeps around an ellipse.

Circular & linear polarisations are special cases of the elliptical one.
A monochromatic vector field is always elliptically polarised.

Absorptive linear polarisers

Most sources of visible light (but not lasers) produce *randomly polarised* (aka *unpolarised*) waves.
This is only possible for non-monochromatic waves.

Linear polarisers: optical filters that only transmit light of a specific polarisation (determined by the transmission axis).

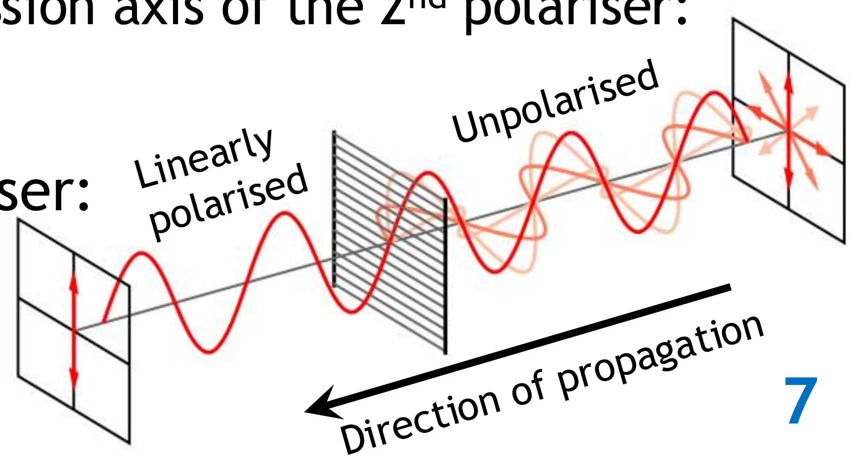
Dichroism (observed in some crystals):
preferential absorption of light polarised in a particular direction.
Used to build *absorptive linear polarisers*.

Two perfect polarisers; angle α between the transmission axes.
Intensity of light after the 1st polariser: I_0.
Components of this light wrt the transmission axis of the 2nd polariser:

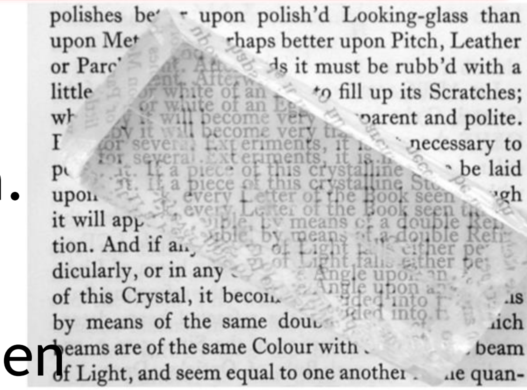
$$\vec{E}_0 = \vec{E}_{||} + \vec{E}_{\perp}$$

Intensity of light passing the 2nd polariser:
(Malus's law)

$$I = I_0 \left(\frac{|\vec{E}_{||}|}{|\vec{E}_0|} \right)^2 = I_0 \cos^2 \alpha$$



Circular polarisers



Birefringence (observed in some crystals): dependence of refractive index, and therefore the speed of light, on the direction of polarisation.

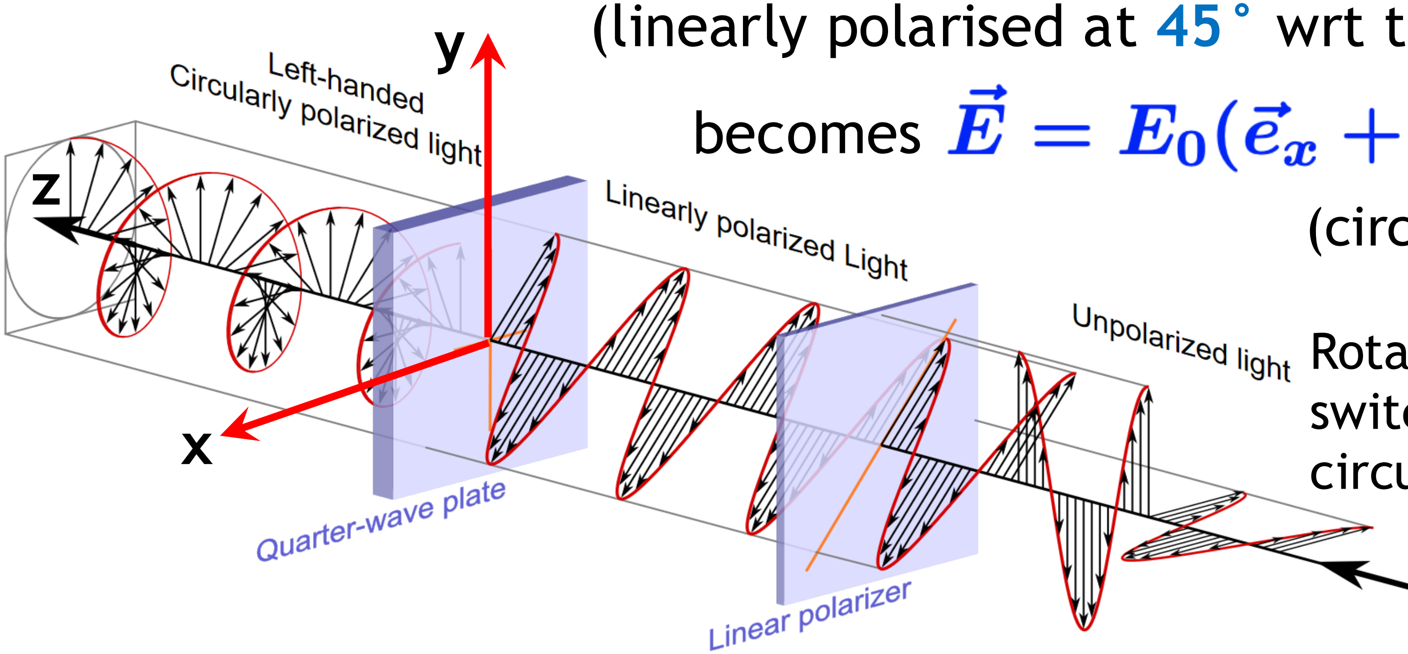
Quarter-wave plate (QWP): introduces a phase difference of $\pi/2$ ("quarter wave") between plane waves with **E** fields aligned with its *slow and fast axes*.

$$\vec{E} = E_0(\vec{e}_x + \vec{e}_y)e^{i(\omega t - kz)}$$

(linearly polarised at **45°** wrt the slow/fast axes)

becomes $\vec{E} = E_0(\vec{e}_x + i\vec{e}_y)e^{i(\omega t - kz)}$

(circularly polarised)



Rotation of QWP by **90°**: switch between left/right circular polarisations

Quarter-wave plates

Computation of QWP thickness (n_1 for fast axis, $n_2 > n_1$ for slow axis).
If **k** is the wavenumber in free space, the fast and slow waves are:

$$E_1(z,t) = E_0 e^{i(\omega t - n_1 k z)}, \quad E_2(z,t) = E_0 e^{i(\omega t - n_2 k z)}$$

Phase difference over a thickness **d**,
for a wave of wavelength λ_0 in free space:

$$(n_2 - n_1)kd = (n_2 - n_1)\frac{2\pi d}{\lambda_0} = \frac{\pi}{2}$$

$$\boxed{\lambda_0 = 4d(n_2 - n_1)}$$

- ❖ Linearly polarised light incident on a QWP:
  - ✓ QWP placed at **0°** or **90°**: light remains linearly polarised;
  - ✓ QWP placed at **45°**: light becomes circularly polarised;
  - ✓ otherwise, light becomes elliptically polarised.
- ❖ QWP can similarly convert circularly polarised light into linearly polarised light.
- ❖ Two QWPs placed with aligned axes (i.e. a *half-wave plate*): conversion of left circularly polarised light to right circularly polarised light, and vice versa.

Summary

❖ Energy flux in a monochromatic plane wave: $\vec{N} = u\vec{v}$

❖ Monochromatic plane waves are always polarised:
- ✓ linearly (the **E** vector remains parallel to a certain axis);
- ✓ circularly (the **E** vector of fixed magnitude sweeps around a circle with an angular frequency **ω**);
- ✓ or elliptically otherwise.

❖ Absorptive linear polarisers made of dichroic materials: wave transmission is described by Malus's law,

$$I = I_0 \left(\frac{|\vec{E}_{||}|}{|\vec{E}_0|} \right)^2 = I_0 \cos^2 \alpha$$

❖ Quarter-wave plates made of birefringent materials: transformation between linear/circular polarisations.