UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 16

EM waves at boundaries of dielectrics

- Boundary conditions for monochromatic waves
- Reflection and refraction
- Critical angle; total internal reflection
- Evanescent fields

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Previous lecture

- Energy flux in a monochromatic plane wave: $\vec{N} = u\vec{v}$

- Monochromatic plane waves are always polarised:
  - ✓ linearly (the **E** vector remains parallel to a certain axis);
  - ✓ circularly (the **E** vector of fixed magnitude sweeps around a circle with an angular frequency **ω**);
  - ✓ or elliptically otherwise.

- Absorptive linear polarisers made of dichroic materials: wave transmission is described by Malus's law,

$$I = I_0 \left(\frac{|\vec{E}_{||}|}{|\vec{E}_0|} \right)^2 = I_0 \cos^2 \alpha$$

- Quarter-wave plates made of birefringent materials: transformation between linear/circular polarisations.

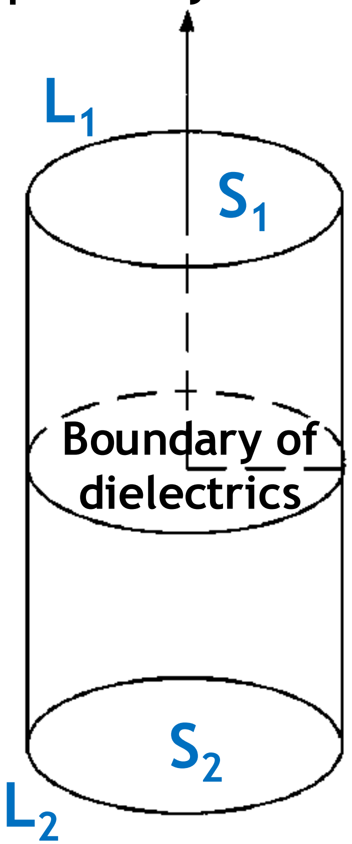
Boundary conditions

Boundaries conditions at boundaries of dielectric materials, in the absence of surface charges and currents (*lectures 8,11*):

$$E_{1t} = E_{2t} \qquad H_{1t} = H_{2t} \qquad \text{(1a,b)}$$

$$D_{1n} = D_{2n} \qquad B_{1n} = B_{2n} \qquad \text{(2a,b)}$$

Consider a pillbox cylinder:



Eq. (M4) in integral form in the absence of free currents:

$$\oint_{L_1} \vec{H}_t d\vec{l} = \int_{S_1} \frac{\partial D_n}{\partial t} dS, \text{ i.e. } \oint_{L_2} \vec{H}_t d\vec{l} = \int_{S_2} \frac{\partial D_n}{\partial t} dS$$

A cylinder of infinitely small height: the condition (1b) for H_t leads to

$$\frac{\partial D_{1n}}{\partial t} = \frac{\partial D_{2n}}{\partial t}$$

For a monochromatic wave, $\frac{\partial \vec{D}}{\partial t} = i\omega \vec{D}$ *(lecture 14)*

Therefore, $D_{1n} = D_{2n}$, i.e. **(2a)** follows from **(1b)**.

Similarly, **(2b)** follows from **(1a)** *[a non-assessed problem]*.

For a monochromatic wave, **only two** independent boundary conditions. We will use (1a,b).

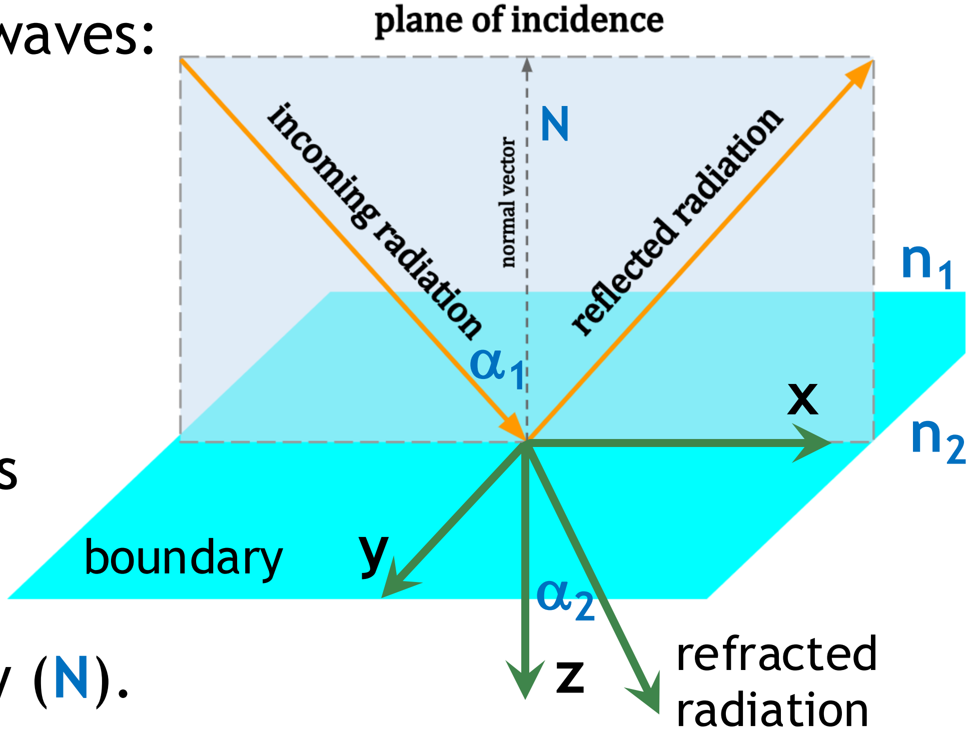
Reflection and refraction (1)

Incident, reflected and refracted waves:

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\vec{r})}$$

$$\vec{E}' = \vec{E}_0' e^{i(\omega' t - \vec{k}'\vec{r})}$$

$$\vec{E}'' = \vec{E}_0'' e^{i(\omega'' t - \vec{k}''\vec{r})}$$



For $\alpha_1 \neq 0$, the *plane of incidence* is defined by the wave vector of the incident wave (**k**) and the normal vector to the boundary (**N**).

Geometric properties of reflection/refraction rely on the *existence* of linear boundary conditions, not on their exact form.

A generic linear boundary condition:

$$A(\vec{r})e^{i\omega t} + B(\vec{r})e^{i\omega' t} + C(\vec{r})e^{i\omega'' t} = 0 \quad \text{at the boundary}$$

This is only possible if $\omega = \omega' = \omega''$

Reflection and refraction (2)

Choose the reference frame so that the boundary is z=0; the x axis is in the plane of incidence (i.e. $k_y=0$). A linear boundary condition:

$$Ae^{-i(k_x x+k_y y)}+Be^{-i(k'_x x+k'_y y)}+Ce^{-i(k''_x x+k''_y y)}=0$$

On the x axis (i.e. y=0), $Ae^{-ik_x x}+Be^{-ik'_x x}+Ce^{-ik''_x x}=0$

This leads to $k_x=k'_x=k''_x$

Similarly considering x=0, we conclude that $k_y=k'_y=k''_y=0$

Incident, reflected and refracted waves *lie in the plane of incidence*.

Definition of wave number:

$$\begin{cases} k^2=(k')^2=\left(\frac{\omega}{v_1}\right)^2=\left(\frac{\omega n_1}{c}\right)^2 \\ (k'')^2=\left(\frac{\omega}{v_2}\right)^2=\left(\frac{\omega n_2}{c}\right)^2 \end{cases}$$

Normal components of wave vectors:

$$\begin{cases} k'_z=-\sqrt{k^2-k_x^2}=-k_z \\ k''_z=\sqrt{(k'')^2-k_x^2}\neq k_z \end{cases}$$

Signs of k_z' and k_z'' are chosen to describe reflection and refraction.

Reflection and refraction (3)

For the **reflected wave**, $k'_x = k_x; \quad k'_y = k_y = 0; \quad k'_z = -k_z$

Law of reflection: wave vector **k'** lies in the plane of incidence, and the reflection angle is equal to the incidence angle ($\alpha_1 = \alpha_1'$).

For the **refracted wave**, there are two cases.

1) If $(k'')^2 \geq k_x^2$, then k''_z is real;

$$k''_x = k_x; \quad k''_y = k_y = 0; \quad k''/k = n_2/n_1$$

The wave vector lies in the plane of incidence, and

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{k_x/k}{k''_x/k''} = \frac{k''}{k} = \frac{n_2}{n_1}$$: **Snell's law of refraction**.

2) The case $(k'')^2 < k_x^2$, i.e. $k_x/k'' > 1$, meaning that

$$\sin\alpha_1 = \frac{k_x}{k} = \frac{k_x}{k''} \cdot \frac{k''}{k} = \frac{k_x}{k''} \cdot \frac{n_2}{n_1} > \frac{n_2}{n_1},$$

which is only possible for $n_2 < n_1$.

Total internal reflection

In this case, $k''_z = \pm i\sqrt{k_x^2 - (k'')^2} = \pm i/h$,

where **h** is the *penetration depth*. Then

$$\vec{E}'' = \vec{E}''_0 e^{i(\omega t - \vec{k}''\vec{r})} = \vec{E}''_0 e^{i(\omega t - k_x x - k''_z z)}$$

$$= \vec{E}''_0 e^{i(\omega t - k_x x \pm iz/h)} = \vec{E}''_0 e^{-z/h} e^{i(\omega t - k_x x)}$$

Physical choice of the sign of k''_z : attenuation of the wave.

No wave propagation along the **z** axis, and no absorption. This is *total internal reflection* (TIR).

The *critical angle* (i.e. maximum angle of incidence for which there is a refracted beam):

$$\sin\alpha_c = \frac{n_2}{n_1}$$

($n_2 < n_1$)

Penetration depth in case of TIR:

$$h = \frac{1}{\sqrt{k_x^2 - (k'')^2}} = \frac{1/k}{\sqrt{(k_x/k)^2 - (k''/k)^2}} = \frac{\lambda_1}{2\pi\sqrt{\sin^2\alpha_1 - (n_2/n_1)^2}}$$

Minimum **h** value: grazing incidence for $n_2 \ll n_1$, $h_{min} = \lambda_1/2\pi$

When approaching the critical angle, $h|_{\alpha_1 \to \alpha_c} \to \infty$

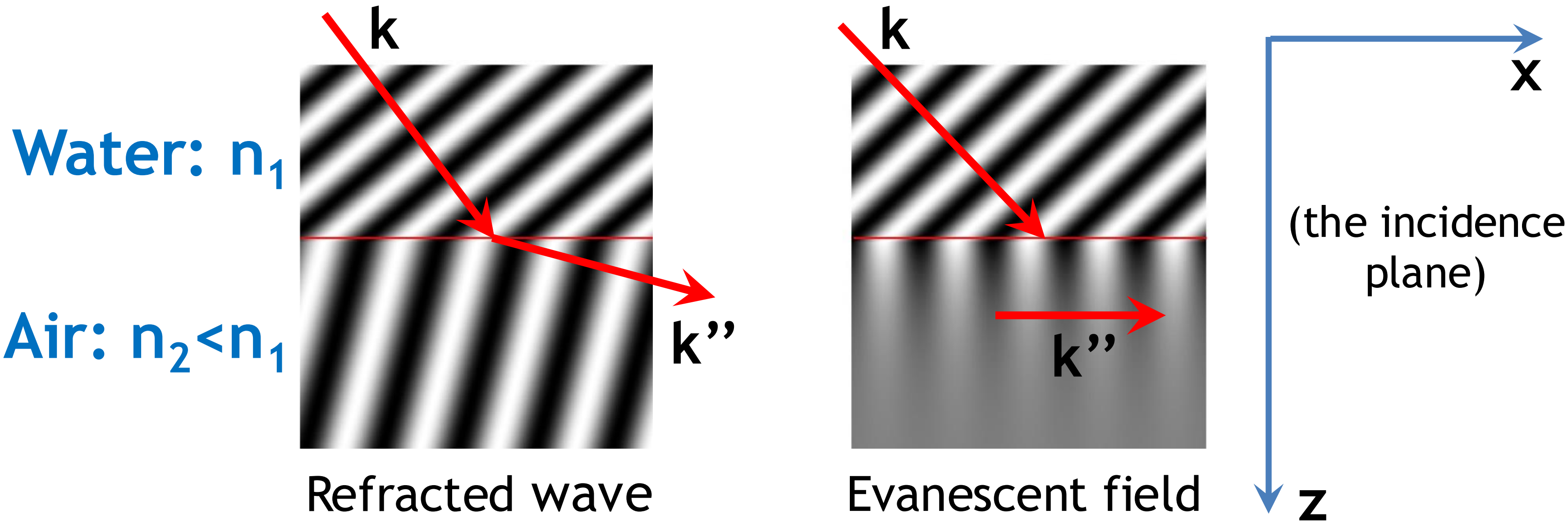
Evanescent fields

$$\vec{E}'' = \vec{E}_0'' \cdot e^{-z/h} \cdot e^{i(\omega t - k_x x)}$$

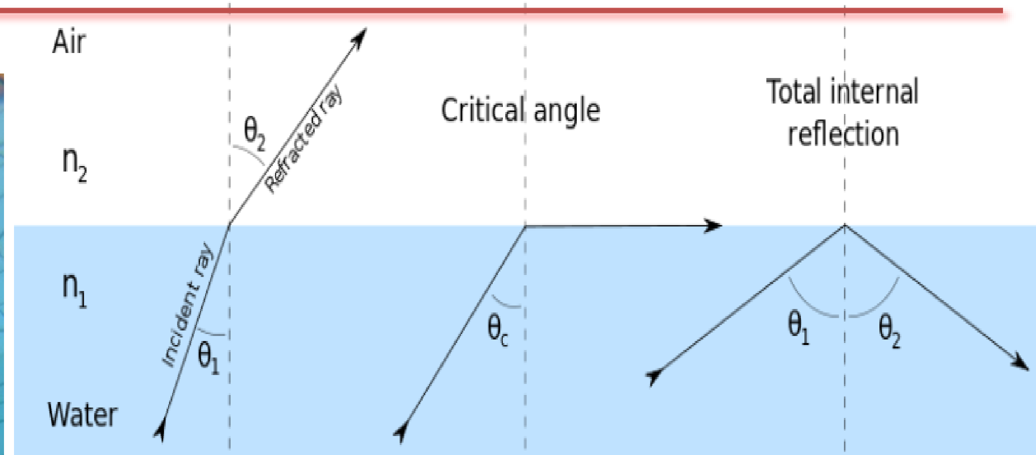
is an example of an *evanescent field*, i.e.
an oscillating EM field located in the vicinity of the source.

Unlike the incident, reflected and refracted waves,
its wave fronts are *perpendicular* to the boundary (i.e. x=const),
and its wave vector is *parallel* to the boundary ($k''=k_x''$).

The existence of the evanescent field is established
fundamentally from the existence of linear boundary conditions.

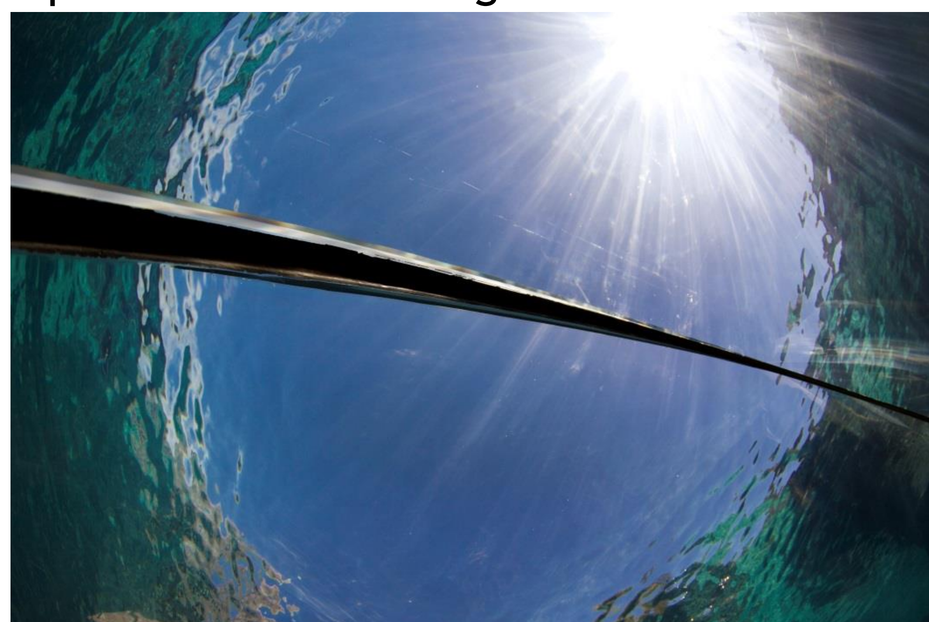


Total internal reflection; Snell's window



"Snell's window"

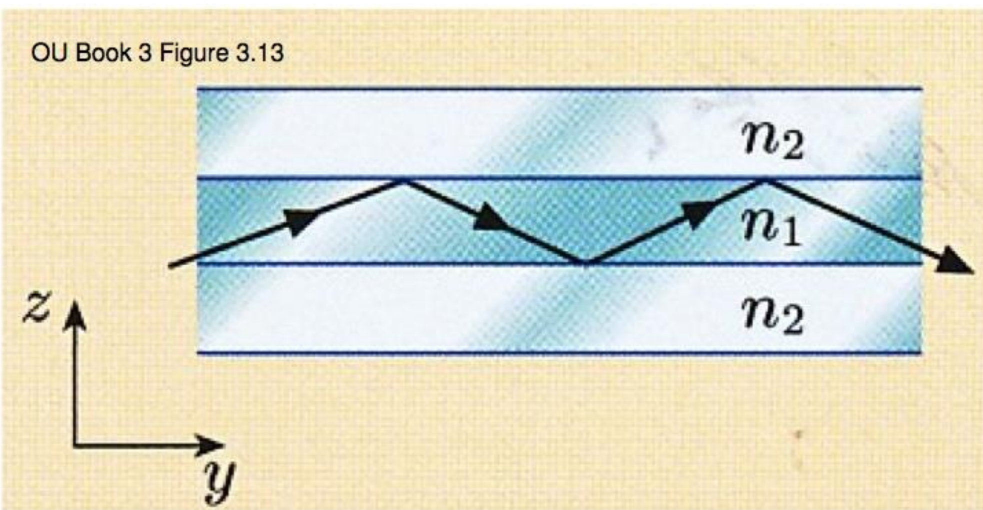
The **180°** angle of view above water is compressed to a **98°** angle of view below water

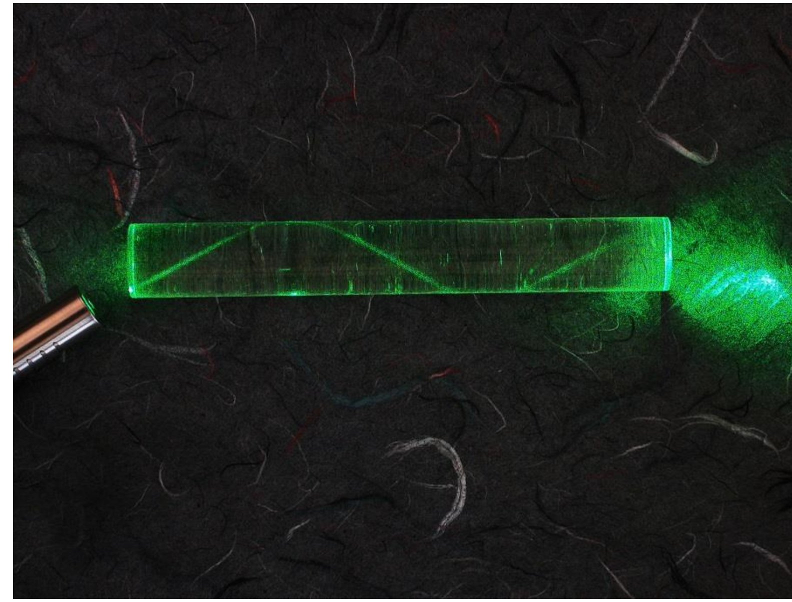


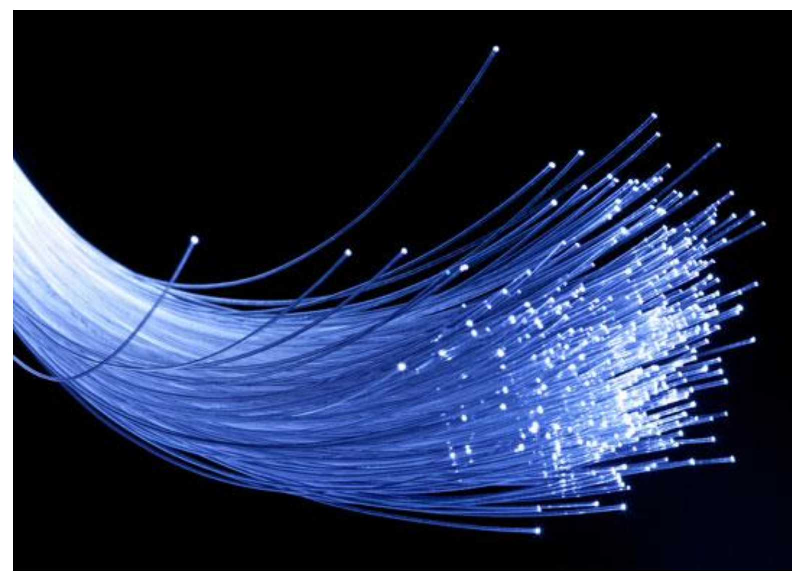
$$\alpha_c = \arcsin(1/1.33) = 48.8^\circ$$

Optical fibres

Optical fibres made of dielectrics with $n_1 > n_2$ can be used to guide EM radiation, e.g. in communications.







Summary

- For monochromatic waves, there are two independent boundary conditions. We will use these:

$$E_{1t} = E_{2t} \qquad H_{1t} = H_{2t}$$

- Geometric laws of reflection and refraction laws follow from the existence of linear boundary conditions. In particular, Snell's law:

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{n_2}{n_1}$$

- Total internal reflection for $n_1 > n_2$ for incidence angles exceeding the critical angle:

$$\sin\alpha_c = \frac{n_2}{n_1}$$

- Total internal reflection gives rise to an evanescent field.