UNIVERSITY OF BIRMINGHAM

Electromagnetism 2 (spring semester 2025)

Lecture 17
Fresnel equations

- S and P polarisations of plane waves
- Derivation of the Fresnel equations
- Main properties of the Fresnel coefficients
- Brewster's angle
- Energy transport at boundary

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Previous lecture

- For monochromatic waves, there are two independent boundary conditions. We will use these:

$$E_{1t} = E_{2t} \qquad H_{1t} = H_{2t}$$

- Geometric laws of reflection and refraction laws follow from the existence of linear boundary conditions. In particular, Snell's law:

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{n_2}{n_1}$$

- Total internal reflection for $n_1 > n_2$ for incidence angles exceeding the critical angle:

$$\sin\alpha_c = \frac{n_2}{n_1}$$

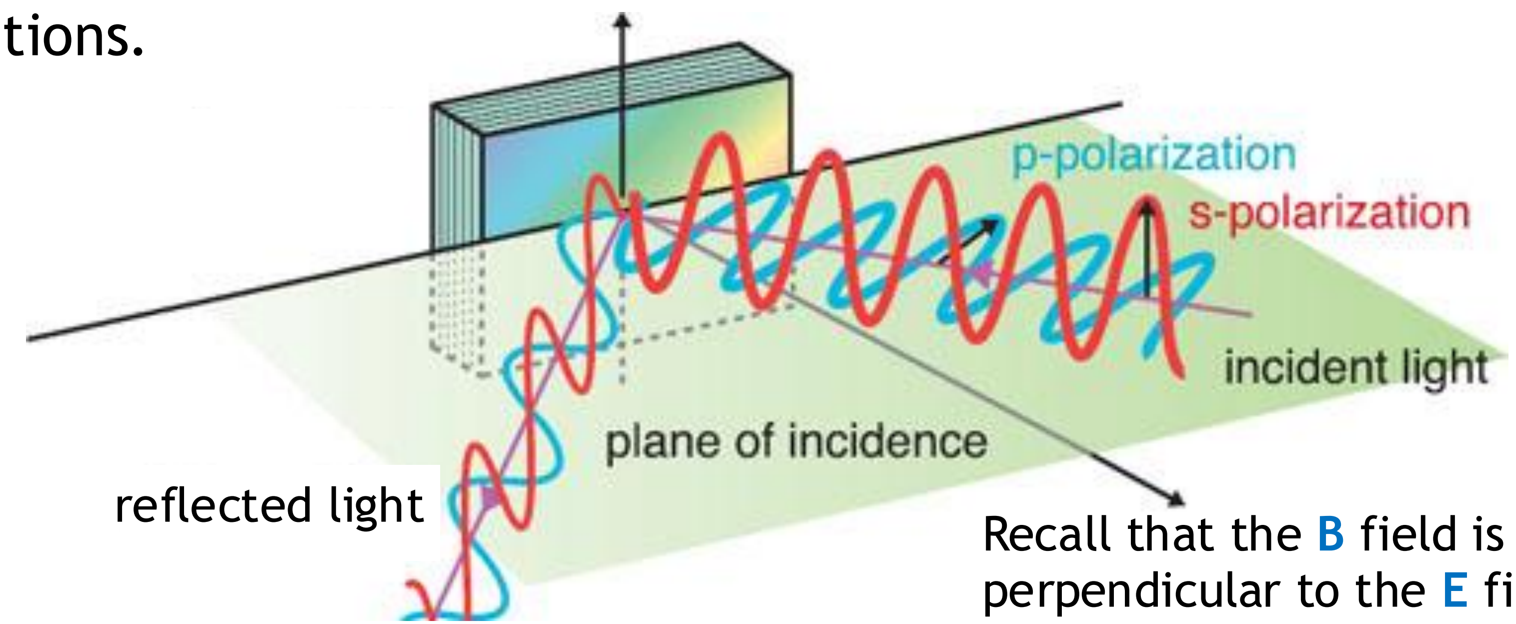
- Total internal reflection gives rise to an evanescent field.

S and P polarisations

The *plane of incidence* contains the incident and reflected **k**-vectors.
(lecture 16)

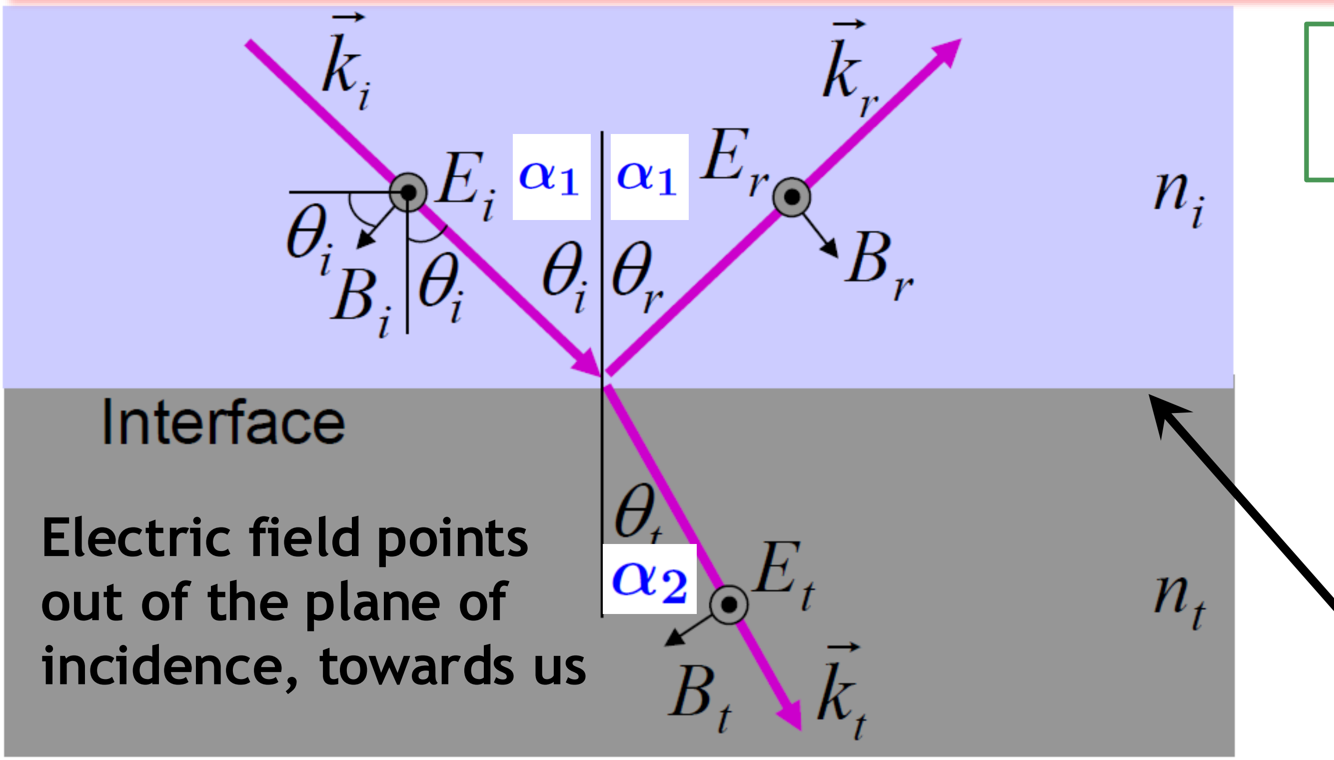
Incident EM wave = superposition of two linearly polarised waves:
1) **E** is perpendicular to the plane of incidence *(lecture 15)*
(*perpendicular,* ⊥, ***s*** *polarisation*);
2) **E** lies in the plane of incidence (*parallel,* ∥, ***p*** *polarisation*);

The fractions of reflected and transmitted energy are different for the two polarisations; they are determined by the boundary conditions.

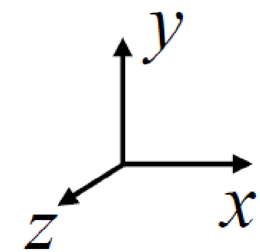


Recall that the **B** field is perpendicular to the **E** field

The S polarisation (1)



In optics, we will assume $\mu_1=\mu_2=1$



Plane of interface: $y=0$

Magnetic field direction is determined using $\vec{B} = \frac{1}{\omega}\left(\vec{k} \times \vec{E}\right)$ *(lecture 14)*

The boundary conditions $E_{1t} = E_{2t};\ B_{1t} = B_{2t}$ lead to the following relations for the field amplitudes:

$$\begin{cases} E + E' = E'' \\ -B\cos\alpha_1 + B'\cos\alpha_1 = -B''\cos\alpha_2 \end{cases}$$

The S polarisation (2)

Using the relation $B = \frac{E}{v} = \frac{nE}{c}$ following from $\vec{B} = \frac{1}{\omega}\left(\vec{k} \times \vec{E}\right)$,

$$\begin{cases} E + E' = E'' \\ \frac{n_1}{c}(-E + E')\cos\alpha_1 = -\frac{n_2}{c}E''\cos\alpha_2 \end{cases}$$

$$n_1(E' - E)\cos\alpha_1 = -n_2(E + E')\cos\alpha_2$$

$$E'(n_1\cos\alpha_1 + n_2\cos\alpha_2) = E(n_1\cos\alpha_1 - n_2\cos\alpha_2)$$

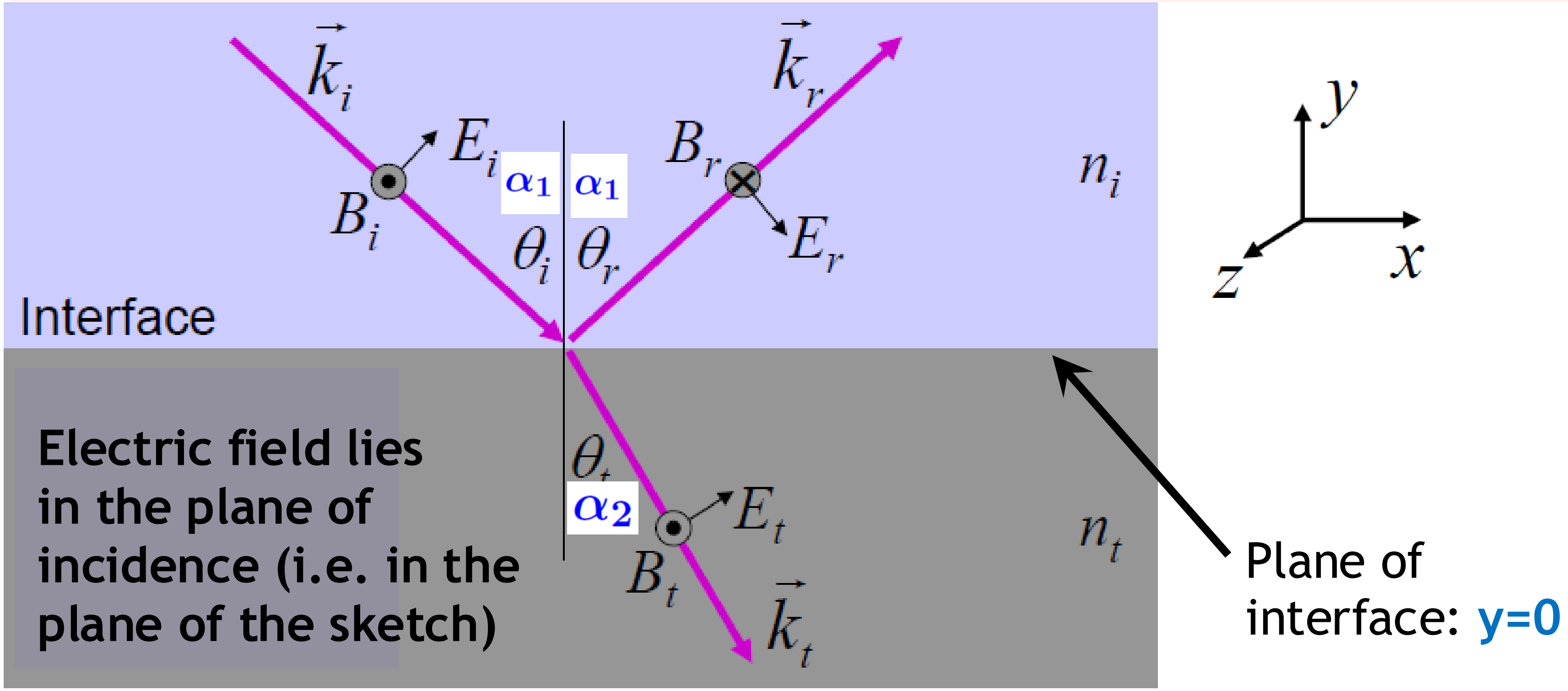
The *reflection coefficient* for the s-polarised (⊥) wave:

$$r_\perp = \frac{E'_\perp}{E_\perp} = \frac{n_1\cos\alpha_1 - n_2\cos\alpha_2}{n_1\cos\alpha_1 + n_2\cos\alpha_2}$$

The *transmission coefficient* for the s-polarised (⊥) wave:

$$t_\perp = \frac{E''_\perp}{E_\perp} = \frac{E_\perp + E'_\perp}{E_\perp} = 1 + r_\perp = \frac{2n_1\cos\alpha_1}{n_1\cos\alpha_1 + n_2\cos\alpha_2}$$

The P polarisation (1)



Note: the **B'** vector points **into the page** due to $\vec{B} = \frac{1}{\omega}\left(\vec{k} \times \vec{E}\right)$

The boundary conditions $E_{1t} = E_{2t};\ B_{1t} = B_{2t}$ lead to

$$\begin{cases} B - B' = B'' \\ E\cos\alpha_1 + E'\cos\alpha_1 = E''\cos\alpha_2 \end{cases}$$

The P polarisation (2)

Using the relation $B = \frac{E}{v} = \frac{nE}{c}$,

$$\begin{cases} n_1(E - E') = n_2 E'' & \text{(multiply by } \cos\alpha_2 \text{)} \\ E\cos\alpha_1 + E'\cos\alpha_1 = E''\cos\alpha_2 & \text{(multiply by } n_2 \text{)} \end{cases}$$

$$n_2(E + E')\cos\alpha_1 = n_1(E - E')\cos\alpha_2$$

$$E'(n_1\cos\alpha_2 + n_2\cos\alpha_1) = E(n_1\cos\alpha_2 - n_2\cos\alpha_1)$$

The *reflection coefficient* for the p-polarised (∥) wave:

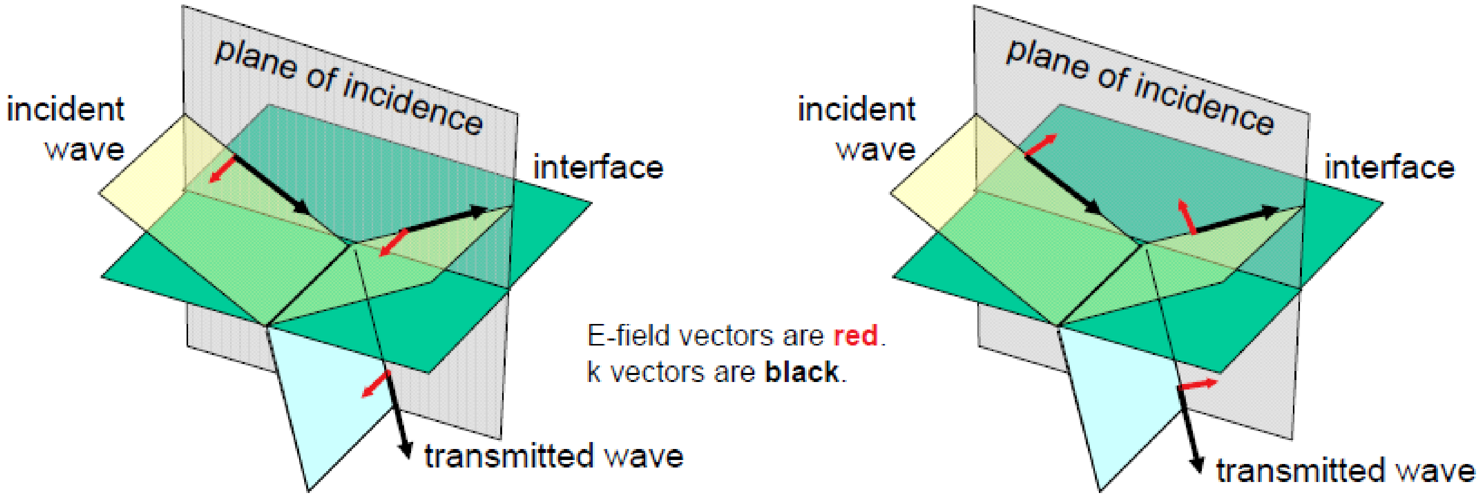
$$r_\parallel = \frac{E'_\parallel}{E_\parallel} = \frac{n_1\cos\alpha_2 - n_2\cos\alpha_1}{n_1\cos\alpha_2 + n_2\cos\alpha_1}$$

The *transmission coefficient* for the p-polarised wave:

$$t_\parallel = \frac{E''_\parallel}{E_\parallel} = \frac{n_1}{n_2}\frac{E_\parallel - E'_\parallel}{E_\parallel} = \frac{n_1}{n_2}(1 - r_\parallel) = \frac{2n_1\cos\alpha_1}{n_1\cos\alpha_2 + n_2\cos\alpha_1}$$

Fresnel equations

for the amplitude reflection and transmission factors



For s-polarised wave,

$$r_\perp = \frac{n_1 \cos\alpha_1 - n_2 \cos\alpha_2}{n_1 \cos\alpha_1 + n_2 \cos\alpha_2}$$

$$t_\perp = \frac{2n_1 \cos\alpha_1}{n_1 \cos\alpha_1 + n_2 \cos\alpha_2}$$

For p-polarised wave,

$$r_\parallel = \frac{n_1 \cos\alpha_2 - n_2 \cos\alpha_1}{n_1 \cos\alpha_2 + n_2 \cos\alpha_1}$$

$$t_\parallel = \frac{2n_1 \cos\alpha_1}{n_1 \cos\alpha_2 + n_2 \cos\alpha_1}$$

For both polarisations, $n_1 \sin\alpha_1 = n_2 \sin\alpha_2$

If the incident wave is S- or P-polarised,
the reflected and refracted waves are also S- or P-polarised

The case $n_1<n_2$ (e.g. air–glass)

A normal incidence ($\alpha_1=\alpha_2=0$), the plane of incidence is not defined;

$$t_{\parallel} = t_{\perp}, \quad p_{\parallel} = p_{\perp}$$

At grazing incidence ($\alpha_1=\pi/2$), *total reflection* for both polarisations:

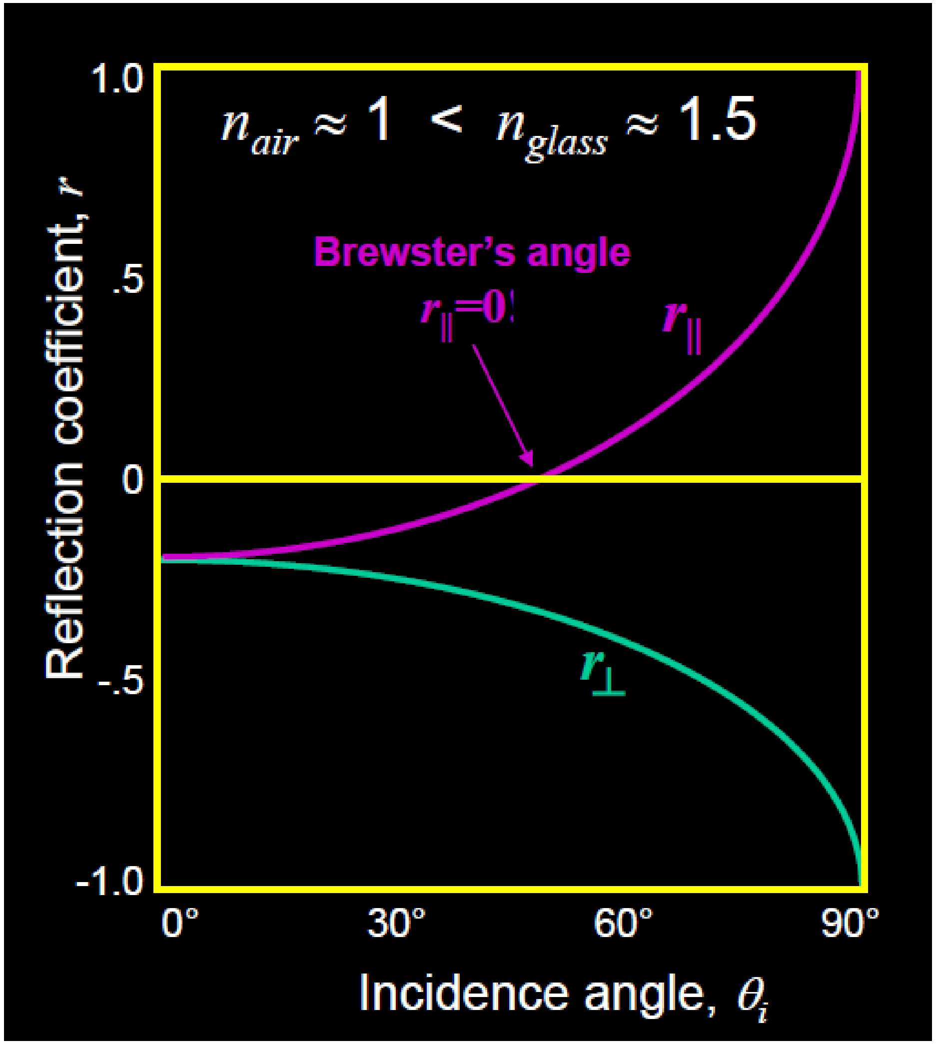
$$t_{\parallel} = t_{\perp} = 0$$

For a wave incident at the *Brewster's angle*, no reflection of p-polarised light:

$$r_{\parallel} = 0$$

The signs of r and t are determined by the choice of the axis directions, and have no physical meaning.

In the plot, $n_1=1$; $n_2=1.5$



The case $n_1>n_2$ (e.g. glass–air)

In the plot, $n_1=1.5$; $n_2=1$

<u>Main features:</u>

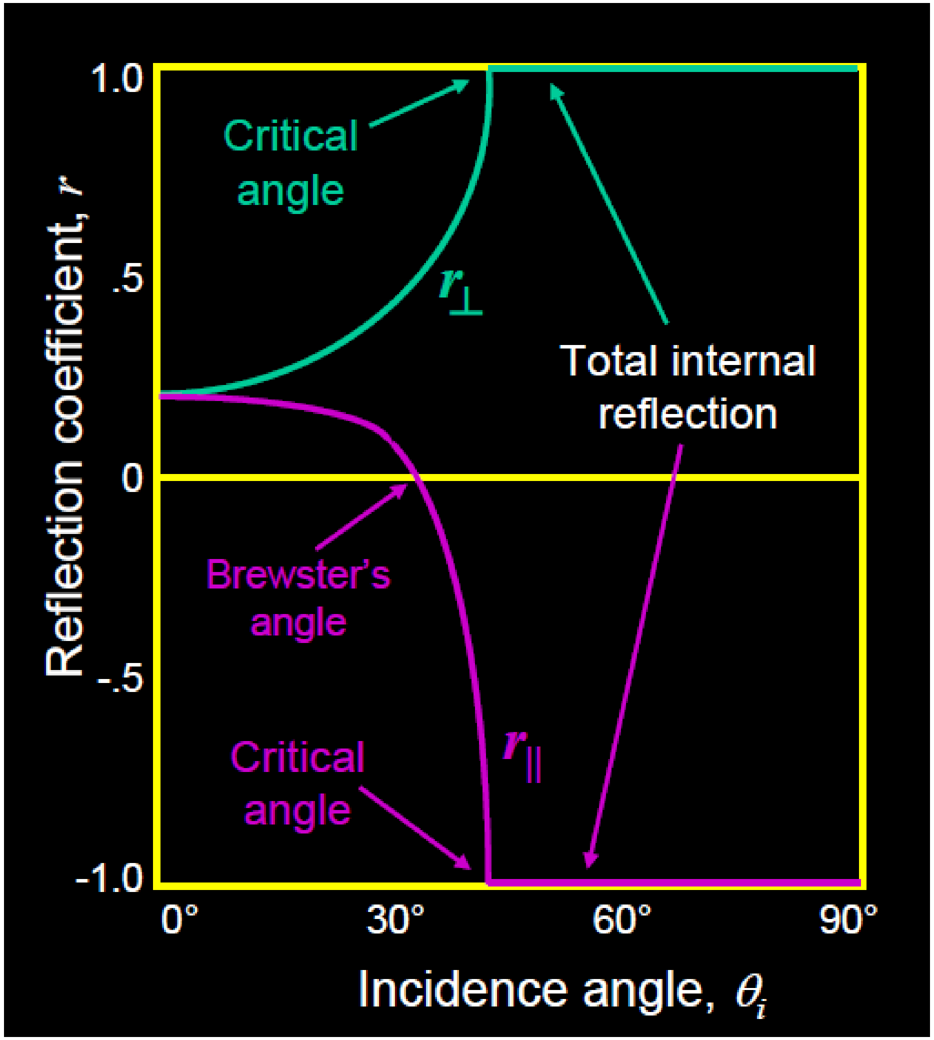
1) Reflection at normal incidence, and the Brewster's angle: same conclusions as in the $n_1<n_2$ case.

2) Total internal reflection above the critical angle of incidence:

$$\sin\alpha_c = \frac{n_2}{n_1}$$

(see also lecture 16)

From Fresnel equations, we obtain $|r|=1$ at $\alpha_2=\pi/2$. This is indeed total reflection.



Brewster's angle: $r_{\parallel} = 0$

$$\begin{cases} n_2 \cos\alpha_1 = n_1 \cos\alpha_2 & \text{(from Fresnel's equations)} \\ n_1 \sin\alpha_1 = n_2 \sin\alpha_2 & \text{(Snell's refraction law)} \end{cases}$$

Multiply the two equations:

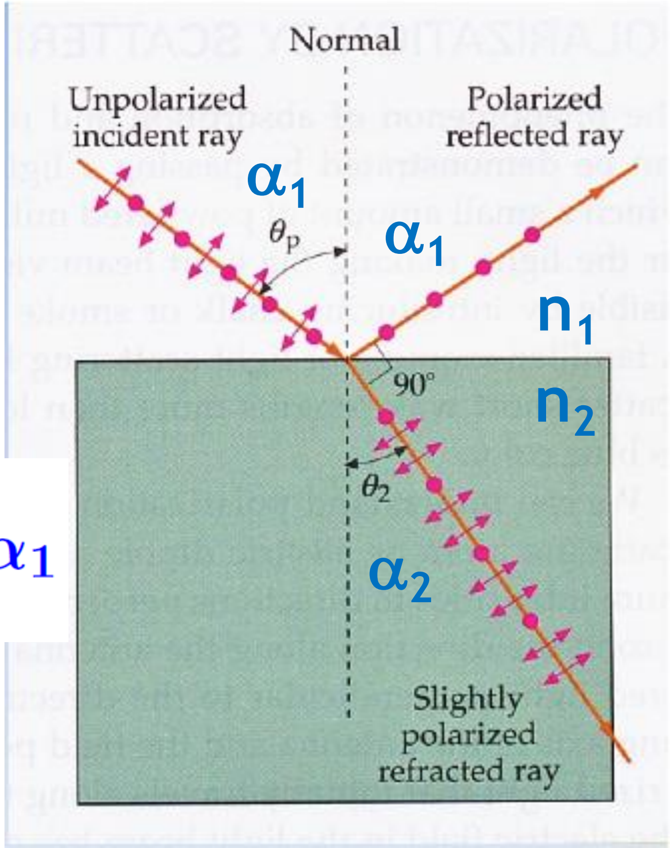
$$n_1 n_2 \sin\alpha_1 \cos\alpha_1 = n_1 n_2 \sin\alpha_2 \cos\alpha_2$$

$$\sin 2\alpha_1 = \sin 2\alpha_2$$

$\alpha_1 = \alpha_2$ (trivial solution, $n_1 = n_2$), or

$$2\alpha_1 = \pi - 2\alpha_2$$

$\alpha_1 + \alpha_2 = \pi/2$ i.e. $\boxed{\vec{k}' \perp \vec{k}''}$



$$n_1 \sin\alpha_1 = n_2 \sin\left(\frac{\pi}{2} - \alpha_1\right) = n_2 \cos\alpha_1$$

Brewster's angle: $\boxed{\tan\alpha_1 = \frac{n_2}{n_1}}$

Energy transport at boundary

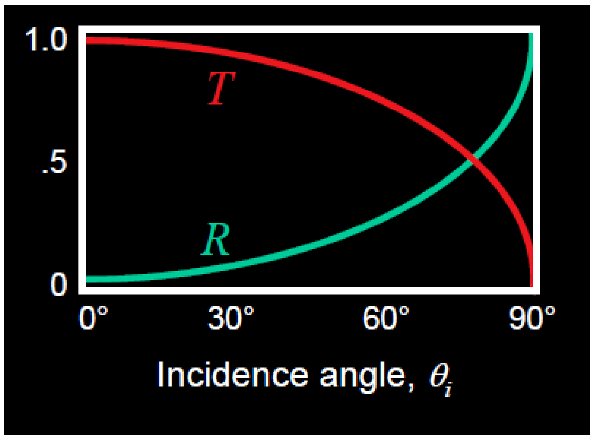
Reflectance (proportion of energy reflected off surface): $R = r^2$

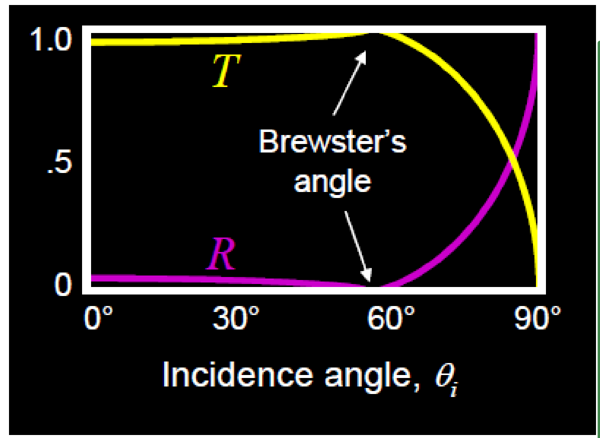
Transmittance (proportion of energy transmitted): $T = 1 - R$

Perpendicular (s) polarisation

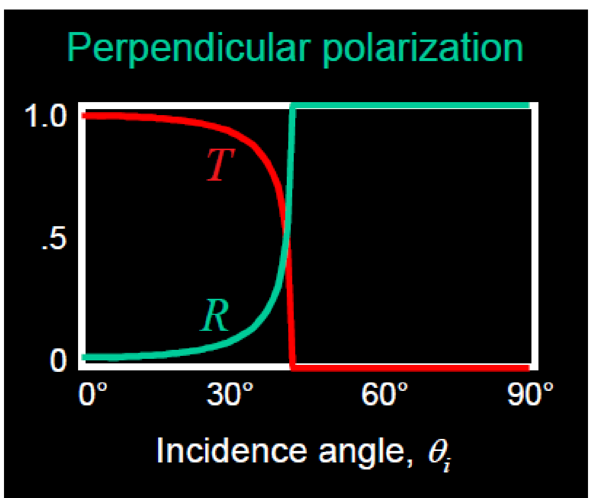
Parallel (p) polarisation

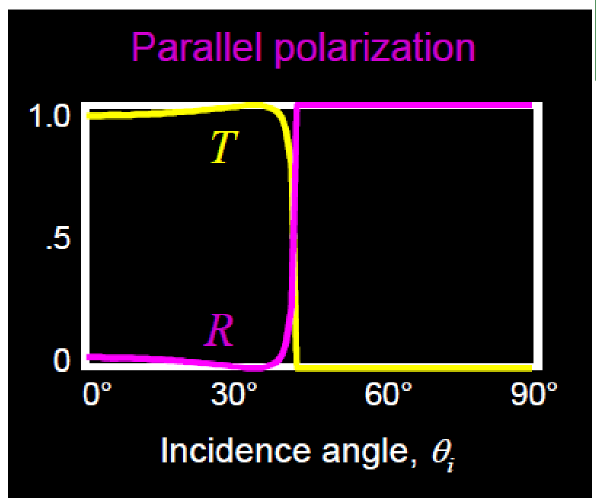
Air to glass





Glass to air





Natural light reflected off an air/dielectric boundary is largely s-polarised, except for very large and very small angles of incidence.

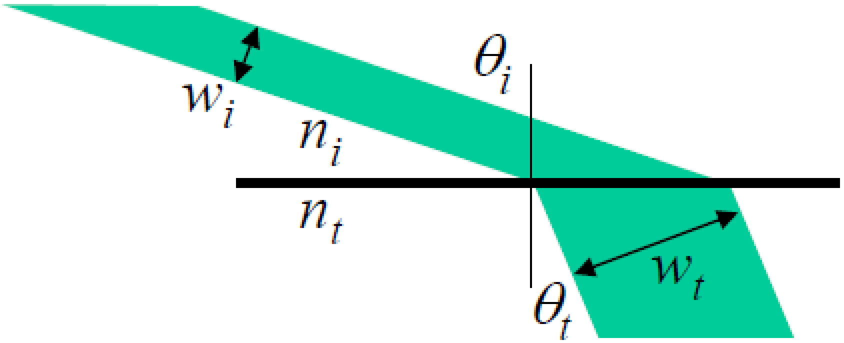
Energy conservation

Reflection and transmission coefficients **r**, **t**
are *ratios of E-field amplitudes*, not ratios of energy fluxes.

Reflectance **R** and the reflection coefficient **r**: $R = r^2$
(the reflected wave propagates in the same medium, at the same angle)

Transmittance **T** and the transmission coefficient **t**:
account for expansion/contraction of the beam on refraction, and
different wave speeds and energy densities (**u~εE^2**) in the two media,

$$T = \left(\frac{n_2 \cos\alpha_2}{n_1 \cos\alpha_1}\right) t^2$$



Fresnel coefficients **R** and **T** satisfy energy conservation,
separately for the **s**- and **p**-polarisations: $R + T = 1$

Reflection at normal incidence

From Fresnel's equations, for $\alpha_1 = \alpha_2 = 0$,

$$R_{\parallel} = R_{\perp} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \quad \text{and} \quad T = 1 - R = \frac{4 n_1 n_2}{(n_1 + n_2)^2}$$

For air-glass interface ($n_1=1$, $n_2=1.5$), $R=4\%$ and $T=96\%$.

For air-water interface,
$R=2\%$ for visible light ($n=1.33$);
$R=64\%$ for radiowaves ($n=9$).

Reflectance at normal incidence does not depend on the polarisation of the wave.

Implications for photography: lens flare.

Examples

Reflected natural light is largely s-polarised, except for very large and very small angles of incidence.

Polariser filters are used in photography to remove reflected sunlight.

Polaroid sunglasses principle:

s-polarised light reflected off a horizontal surface is polarised mainly *horizontally*.

Summary

Dynamic properties of reflection at the boundary of dielectrics.

- The reflection and transmission (Fresnel) coefficients are different for **p** (∥) and **s** (⊥) polarised waves.
- Reflectance and transmittance at normal incidence:

$$R_{\|} = R_{\perp} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \qquad T = 1 - R = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

- For grazing incidence, $R_{\|} = R_{\perp} = 1$.
- For incidence at the Brewster's angle $\alpha_B = \arctan\left(\frac{n_2}{n_1}\right)$,

  the **p**-polarised wave is fully transmitted: $R_{\|} = 0$.