UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 18
Dispersion of EM waves in dielectrics

- Classical theory of dispersion
- Complex refractive index
- Propagation and absorption of waves in dielectrics

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Previous lecture

Dynamic properties of reflection at the boundary of dielectrics.

- The reflection and transmission (Fresnel) coefficients are different for **p** (∥) and **s** (⊥) polarised waves.
- Reflectance and transmittance at normal incidence:

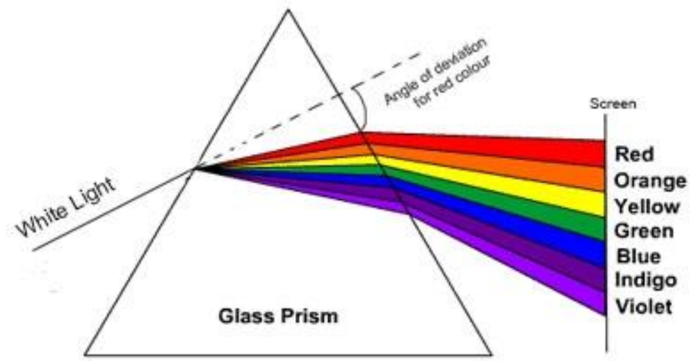
$$R_{||} = R_{\perp} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \qquad T = 1 - R = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

- For grazing incidence, $R_{||} = R_{\perp} = 1$.
- For incidence at the Brewster's angle $\alpha_B = \arctan\left(\frac{n_2}{n_1}\right)$,

  the **p**-polarised wave is fully transmitted: $R_{||} = 0$.

Dispersion of EM waves

- *Dispersion*: dependence of refractive index on the frequency of EM waves, $n=n(\omega)$, occurring in all media.



- Dispersion of radiowaves is observed in interstellar medium.

- Dispersion is due to oscillations of charges driven by the oscillating electromagnetic field.

- Classical theory of dispersion in dielectrics (developed by Lorentz): damped oscillations of bound charges.

- Complete theory is based on quantum mechanics. However classical theory leads to many correct results.

Electron in an oscillating EM field

Consider a single *bound electron*, and a field $\vec{E} = \vec{e}_x E_0 e^{i\omega t}$

The effect of the magnetic field in an EM wave is negligible:

$$F_B / F_E < \frac{q v_{\text{electron}} B}{qE} = v_{\text{electron}} \frac{B}{E} = \frac{v_{\text{electron}}}{v_{\text{wave}}} \ll 1$$

Driven damped harmonic oscillator model:

assuming oscillation amplitude ≪ wavelength, i.e. neglecting *spatial dispersion*,

$$m\ddot{x} = \underbrace{-qE_0 e^{i\omega t}} - \underbrace{m\gamma\dot{x}} - \underbrace{m\omega_0^2 x}$$

Electric force.
$q>0$: elementary charge.

Dissipative (damping) term:
collisions, radiation.

Restoring force.
ω_0: natural resonant frequency.

Let's look for a steady solution: $x(t) = x_0 e^{i\omega t}$

$$m(i\omega)^2 x(t) = -qE(t) - m\gamma \cdot i\omega x(t) - m\omega_0^2 x(t)$$

$$qE(t) = m\left[\omega^2 - i\gamma\omega - \omega_0^2\right] \cdot x(t)$$

$$x(t) = \frac{q/m}{(\omega^2 - \omega_0^2) - i\gamma\omega} E(t)$$

x(t) and E(t) oscillate at the same frequency, out of phase

Magnitude of oscillations

Magnitude of oscillations:

$$x_0 = \frac{qE_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} =$$

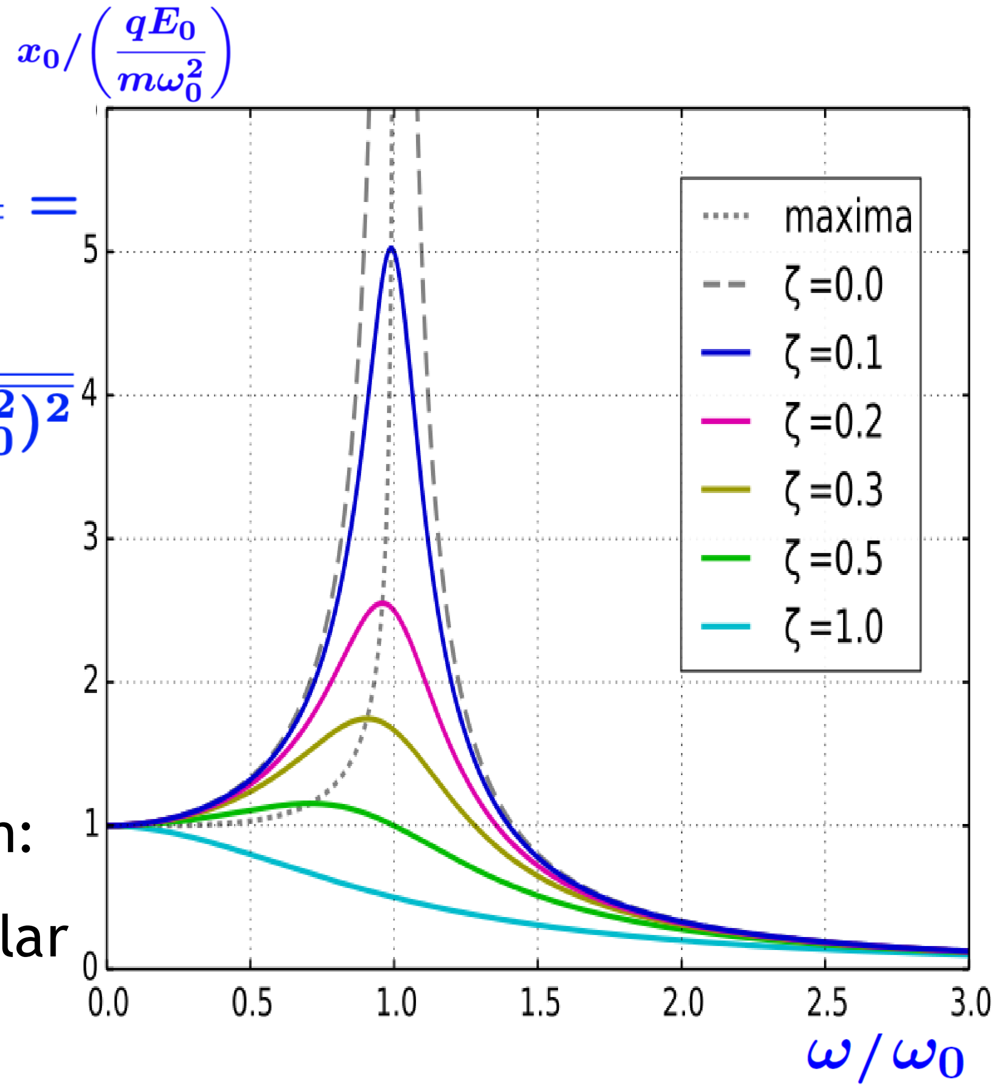
$$\frac{qE_0}{m\omega_0^2} \frac{1}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + (\gamma\omega/\omega_0^2)^2}}$$

The mean dissipated power, $q\langle E(t) \cdot \dot{x}(t) \rangle$, also has a maximum at $\omega \approx \omega_0$

Quantum mechanical interpretation:

$\hbar\omega_0$: energy of atomic/molecular transition

$1/\gamma$: lifetime of an excited energy state

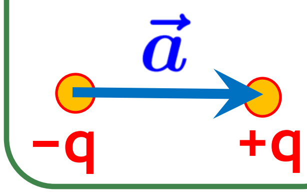


The *damping ratio* is defined as $\xi = \gamma/(2\omega_0)$

Complex relative permittivity

Oscillating polarisation of the dielectric material
(= dipole moment per unit volume, *lecture 7*):

$$\vec{p} = q\vec{a}$$



$$P(t) = -n_e q x(t) = \frac{n_e q^2/m}{(\omega_0^2 - \omega^2) + i\gamma\omega} E(t)$$

n_e: density of electrons [1/m^3]

Relative permittivity (*lecture 8*):

$$\varepsilon = 1 + \chi_E = 1 + \frac{P}{\varepsilon_0 E} = 1 + \frac{n_e q^2/m\varepsilon_0}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

Finally, $\varepsilon = 1 + \frac{\omega_P^2}{(\omega_0^2 - \omega^2) + i\gamma\omega}$,

Recall that
$n = \sqrt{\varepsilon\mu} \approx \sqrt{\varepsilon}$

where $\omega_P = \sqrt{\frac{n_e q^2}{m_e \varepsilon_0}}$ is the *plasma frequency* (depends on n_e only).

- Dispersion: ε and n depend on the frequency ω of the wave.
- Absorption: ε and n are *complex*, due to the damping term.

Relative permittivity near resonance

Complex relative permittivity:

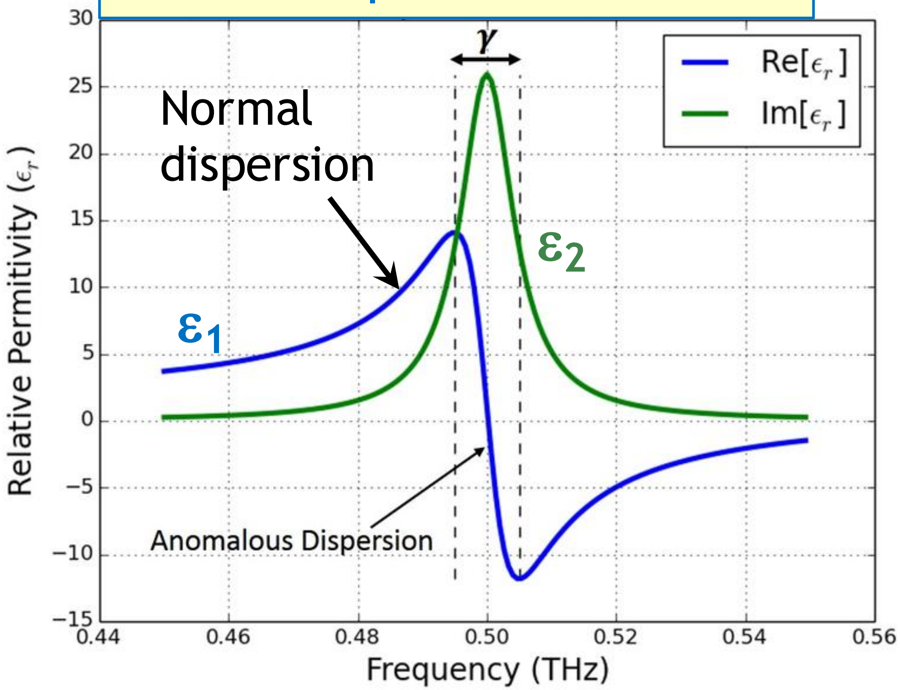
$$\varepsilon(\omega) = 1 + \frac{\omega_P^2}{(\omega_0^2 - \omega^2) + i\gamma\omega} = \varepsilon_1 - i\varepsilon_2$$

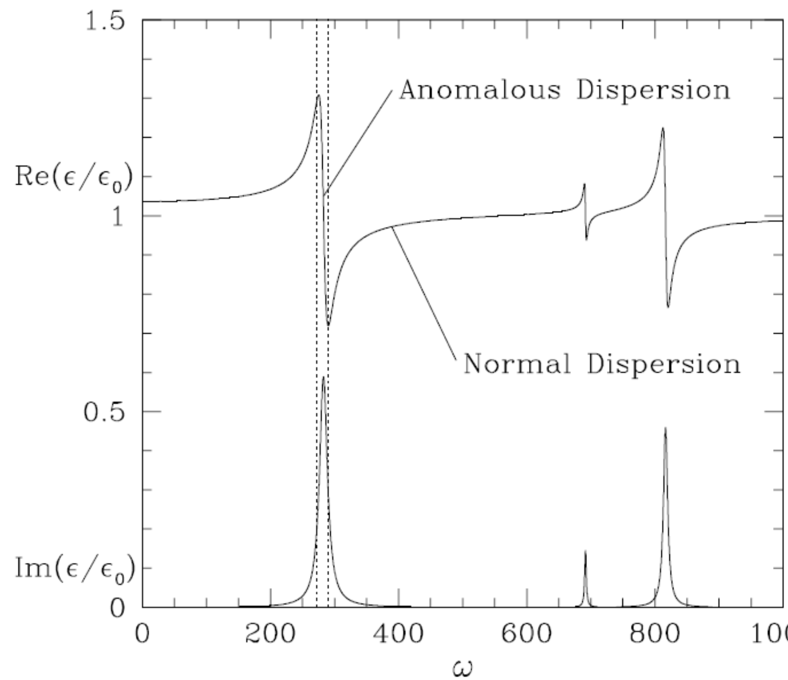
Real and imaginary parts, written explicitly:

$$\varepsilon_1(\omega) = 1 + \frac{\omega_P^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$\varepsilon_2(\omega) = \frac{\omega_P^2 \cdot \gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

Relative permittivity near an absorption resonance





Refractive index vs frequency

Behaviour of the functions $\varepsilon_{1,2}(\omega)$:

- $\varepsilon_1(\omega)\approx 1$ far from resonance; $\varepsilon_1(\omega_0)=1$; $\varepsilon_1(\omega)$ falls across resonance.
- $\varepsilon_2(\omega)\approx 0$ far from resonance; $\varepsilon_2(\omega)$ reaches a maximum near the resonance; $\varepsilon_2(\omega_0)=\omega_P^2/(\gamma\omega_0)$.

Complex refractive index $n=\sqrt{\varepsilon}$: find real and imaginary parts.

$$n^2=(n_1-in_2)^2=\varepsilon=\varepsilon_1-i\varepsilon_2$$

$$n_1^2-2in_1n_2-n_2^2=\varepsilon_1-i\varepsilon_2$$

$$\begin{cases} n_1^2-n_2^2=\varepsilon_1 \\ 2n_1n_2=\varepsilon_2 \end{cases}$$

Assuming $n_2 \ll n_1$,

$$n_1=\sqrt{\varepsilon_1} \quad \text{and} \quad n_2=\frac{\varepsilon_2}{2\sqrt{\varepsilon_1}}$$

- Dispersion (n_1) and absorption (n_2) are inextricably linked.
- Dielectric materials have multiple absorption bands, corresponding to multiple resonant frequencies.
- In addition to electronic modes discussed above, vibrational and rotational modes are often present.

Wave propagation in dielectrics

Considering $n = n_1 - in_2$ and $k = \frac{\omega}{v} = \frac{n\omega}{c}$,

a plane monochromatic wave propagating in the +z direction is

$$\vec{E} = \vec{E}_0 e^{i(\omega t - kz)} = \vec{E}_0 e^{i\omega(t - nz/c)}$$
$$= \underbrace{\vec{E}_0 e^{i\omega(t - n_1 z/c)}}_{\text{Plane wave propagating with a speed } v=c/n_1} \underbrace{e^{-\omega n_2 z/c}}_{\text{Exponentially decreasing amplitude: absorption}}$$

Time-averaged energy flux: $\langle N(z) \rangle \sim \langle |E(z)|^2 \rangle \sim e^{-2\omega n_2 z/c}$

Absorption coefficient: $K=2\omega n_2/c$.

Characteristic *attenuation length* L is defined by

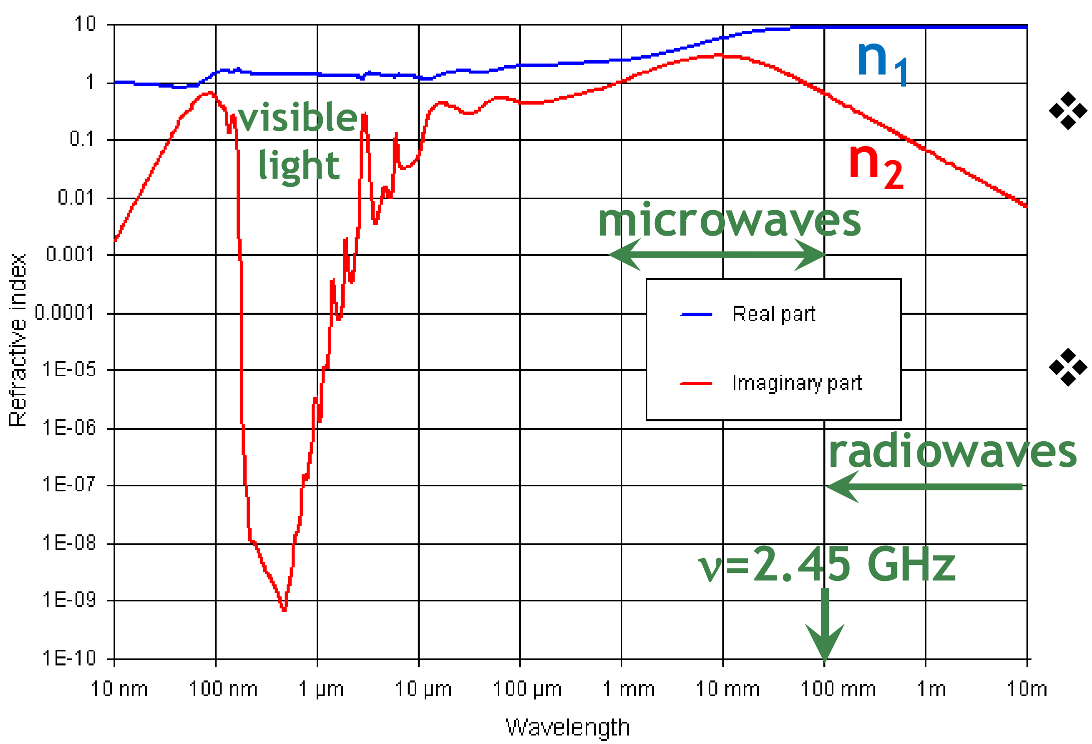
$$\langle N(L) \rangle / \langle N(0) \rangle = e^{-2\omega n_2 L/c} = \frac{1}{2}$$

Therefore (denoting wavelength in free space λ_0),

$$L = \frac{\ln 2}{2} \frac{c}{n_2 \omega} \approx \frac{c}{n_2 \omega} = \frac{n_1 v}{n_2 \omega} = \frac{n_1}{n_2 k} = \frac{n_1 \lambda}{2\pi n_2} = \frac{\lambda_0}{2\pi n_2}$$

Water: a polar dielectric

- Low frequencies ($\omega \ll 10^{10}$ Hz): no resonances, no dispersion and negligible absorption. Static limit: $\varepsilon = 81; \quad n \approx \sqrt{\varepsilon} = 9$.
- Microwaves ($\omega \sim 10^{11}$ Hz): absorption bands due to rotational states.
- Infrared ($\omega = 10^{13} - 10^{14}$ Hz): absorption bands due to vibrational states.
- Visible light ($\omega = 4 - 8 \times 10^{14}$ Hz): negligible absorption, transparency.
- Ultraviolet ($\omega = 10^{15} - 10^{16}$ Hz): large absorption due to plasmons.
- Higher frequencies: negligible absorption.



- At $\nu = 2.45$ GHz (i.e. $\lambda_0 = 12$ cm), $n_2 = 0.5$, and the absorption length is $L \approx \lambda_0 / (2\pi n_2) \approx 4$ cm.
- This is the frequency typically used in microwave ovens.

Summary

- Dispersion (i.e. dependence of **n** on **ω**) arises from oscillations of charges driven by an oscillating electric field.

- Classical theory of dispersion for a single resonant frequency ω_0 leads to a complex relative permittivity varying with frequency:

$$\varepsilon = 1 + \frac{\omega_P^2}{(\omega_0^2 - \omega^2) + i\gamma\omega} \quad \text{with} \quad \omega_P = \sqrt{\frac{n_e q^2}{m_e \varepsilon_0}}$$

and a complex refractive index $n = \sqrt{\varepsilon}$.

- The complex part of refractive index (n_2) describes the absorption of EM waves near a resonance frequency.

- Wave attenuation length in dielectrics: $L \approx \dfrac{\lambda_0}{2\pi n_2}$.