UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 19
Low-frequency EM waves in conductors

- Ohm's law; conductivity in the low-frequency regime
- Wave number in conductors
- Definition of good conductors
- Refractive index in good conductors

Prof. Evgueni Goudzovski • Physics West 212 • goudzovs@cern.ch

Previous lecture

❖ Dispersion (i.e. dependence of n on ω) arises from oscillations of charges driven by an oscillating electric field.

❖ Classical theory of dispersion for a single resonant frequency ω_0 leads to a complex relative permittivity varying with frequency:

$$\varepsilon = 1 + \frac{\omega_P^2}{(\omega_0^2 - \omega^2) + i\gamma\omega} \quad \text{with} \quad \omega_P = \sqrt{\frac{n_e q^2}{m_e \varepsilon_0}}$$

and a complex refractive index $n = \sqrt{\varepsilon}$.

❖ The complex part of refractive index (n_2) describes the absorption of EM waves near a resonance frequency.

❖ Wave attenuation length in dielectrics: $L \approx \dfrac{\lambda_0}{2\pi n_2}$.

Free electrons in conductors

Consider a *free electron* in conductor, and a field $\vec{E} = \vec{e}_x E_0 e^{i\omega t}$

Equation of motion: $m\ddot{x} = \underbrace{-qE_0 e^{i\omega t}}_{\text{Electric force}} \underbrace{- m\gamma\dot{x}}_{\text{Dissipative term}}$

Electric force.
q>0: elementary charge.

Dissipative (damping) term:
collisions with lattice.

Difference to dielectrics (*lecture 18*): there is no restoring force.

The damping constant γ: mean frequency of collisions with lattice.

Mean time between collisions: $\tau_c = 1/\gamma$

For copper,
$\tau_c = 3 \times 10^{-14}$ s

Look for a steady solution: $x(t) = x_0 e^{i\omega t}$

$$m(i\omega)^2 x(t) = -qE(t) - m\gamma \cdot i\omega x(t)$$

$$x(t) = \frac{q/m}{\omega^2 - i\omega\gamma} \cdot E(t)$$

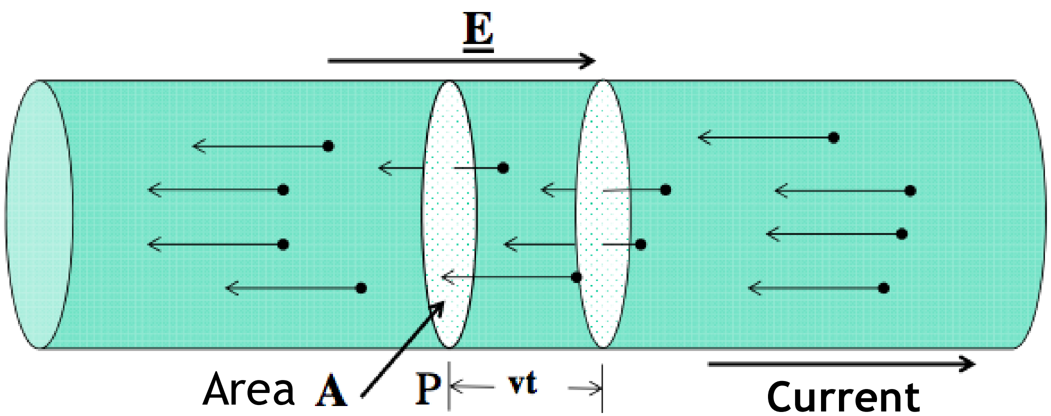
Electron speed:

$$v(t) = \dot{x}(t) = i\omega x(t) = -\frac{q/m}{\gamma + i\omega} \cdot E(t)$$

A simple model for conductivity

Assumptions:

- ❖ Conduction is due to free electrons not bound to atoms.
- ❖ Electron motion in the direction opposite to the **E**-field is randomised by collisions with the lattice: constant *drift speed*.
- ❖ Further details: the *Drude model* based on the kinetic theory.



Density of free electrons: n_e.
Electron drift speed: v.
Cross-section area: A.

Over the time t, all electrons within the distance vt to the right of the plane P will traverse the plane.

Electric current: $$I = \frac{Q}{t} = -\frac{n_e e A v t}{t} = -n_e e A v$$

Current density: $$\vec{j} = -n_e e \vec{v}$$

($e>0$: elementary charge, replacing previous notation q)

Ohm's law; conductivity

Ohm's law: $\boxed{\vec{j} = \sigma \vec{E}} = -n_e e \vec{v}$

Here σ is the *electrical conductivity* [$\Omega^{-1}m^{-1}$].
The quantity $\rho=1/\sigma$ is the *electrical resistivity*.

Therefore, $\sigma = -n_e e \frac{v}{E} = n_e e \frac{e/m_e}{\gamma + i\omega} = \boxed{\frac{n_e e^2}{m_e(\gamma + i\omega)}}$

Physical meaning of the complex conductivity:
current density **j(t)** is *out of phase* with the electric field **E(t)**.

The **low-frequency regime** ($\omega \ll \gamma = 1/\tau_c$) is of practical interest.

- ✓ For copper, applies to radiowaves and microwaves (**λ>1 mm**).
- ✓ We will consider the low-frequency case in *lectures 19,20*.
- ✓ Many collisions of an electron with lattice per wave cycle.

The *static conductivity* is real and does not depend on frequency:

$$\sigma = \frac{n_e e^2}{m_e \gamma} = \frac{n_e e^2}{m_e} \tau_c$$

Maxwell's equations in conductors

Assume absence of free charges: ρ=0. *(always the case for ideal conductors, lecture 3)*

Ohm's law for conductors: $\vec{j} = \sigma \vec{E}$.

Assume a LIH material: $\vec{D} = \varepsilon_0 \varepsilon \vec{E}$; $\vec{B} = \mu_0 \mu \vec{H}$.

(M1) $\nabla \vec{E} = 0$	**(M3)** $\nabla \times \vec{E} = -\mu_0 \mu \dfrac{\partial \vec{H}}{\partial t}$
(M2) $\nabla \vec{H} = 0$	**(M4)** $\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon_0 \varepsilon \dfrac{\partial \vec{E}}{\partial t}$

Equations for **E** and **H** fields separately: take curl of **(M3)**

$$-\nabla \times (\nabla \times \vec{E}) = \mu_0 \mu \frac{\partial}{\partial t} \left(\nabla \times \vec{H} \right) = \mu_0 \mu \sigma \frac{\partial \vec{E}}{\partial t} + \varepsilon_0 \varepsilon \mu_0 \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

On the other hand, $\underbrace{\nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \vec{E}) - \nabla^2 \vec{E}}_{\text{See lecture 2}} = -\nabla^2 \vec{E}$

Wave number in conductors

Therefore

(the equation for the H field is identical)

$$\nabla^2 \vec{E} = \underbrace{\sigma \mu_0 \mu \frac{\partial \vec{E}}{\partial t}}_{\text{Conduction term}} + \underbrace{\varepsilon_0 \varepsilon \mu_0 \mu \frac{\partial^2 \vec{E}}{\partial t^2}}_{\text{Displacement term}} \quad (1)$$

- Poor conductor, "small σ": neglect the conduction term; we obtain the *wave equation* valid also in dielectrics *(lecture 13)*.
- Good conductor, "large σ": neglect the displacement term; we obtain the *heat equation*.

Let's look for a plane wave solution, $\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\vec{r})}$

Substitute into Eq. (1), assume $\varepsilon=\mu=1$, use the shortcuts *(lecture 14)*,

$$\nabla \longrightarrow -i\vec{k} \; ; \; \frac{\partial}{\partial t} \longrightarrow i\omega$$

$$-k^2 \vec{E} = i\sigma\mu_0\omega\vec{E} - \varepsilon_0\mu_0\omega^2\vec{E}$$

Recall that $c = 1/\sqrt{\varepsilon_0 \mu_0}$

Finally, the *dispersion relation* is

$$k^2 = \frac{\omega^2}{c^2} - i\mu_0\omega\sigma$$

Physical meaning of the *complex wave number*: attenuation of waves.

Poor and good conductors

$$k^2 = \frac{\omega^2}{c^2} - i\mu_0\omega\sigma$$

Ratio of magnitudes of displacement and conduction terms in the expression for k^2:

$$Q = \frac{\omega^2/c^2}{\mu_0\omega\sigma} = \frac{\omega\varepsilon_0}{\sigma}$$

Recall that $c = 1/\sqrt{\varepsilon_0\mu_0}$

Poor conductor: $Q \gg 1$, i.e. $\omega \gg \sigma/\varepsilon_0$.
The wave number is almost real: $k \approx \omega/c$ (similarly to vacuum). Waves propagate over large distances.

Good conductor: $Q \ll 1$, i.e. $\omega \ll \sigma/\varepsilon_0$.
Copper: $\sigma/\varepsilon_0 \sim 10^{18}\ s^{-1} \gg \gamma = 1/\tau_c \sim 10^{13}\ s^{-1}$, i.e. $Q \ll 1$ is satisfied automatically in the low-frequency regime ($\omega \ll \gamma = 1/\tau_c$).

In this case, $k^2 \approx -i\mu_0\omega\sigma = e^{-i\pi/2}\mu_0\omega\sigma$

Therefore, $k = e^{-i\pi/4}\sqrt{\mu_0\sigma\omega} = \frac{1-i}{\sqrt{2}}\sqrt{\mu_0\sigma\omega}$

Real and imaginary parts are equal:

$k = k_r - ik_i$, with $k_r = k_i = \sqrt{\frac{\mu_0\sigma\omega}{2}}$

Summary

❖ Ohm's law: $\vec{j} = \sigma \vec{E}$

❖ Static conductivity, applies to the low-frequency regime ($\omega \ll \gamma = 1/\tau_c$):

$$\sigma = \frac{n_e e^2}{m_e} \tau_c$$

❖ Complex wave number in conductors, leading to attenuation of EM waves:

$$k^2 = \frac{\omega^2}{c^2} - i\mu_0 \omega \sigma$$

❖ Ratio of displacement/conduction terms:

$$Q = \frac{\omega^2/c^2}{\mu_0 \omega \sigma} = \frac{\omega \varepsilon_0}{\sigma}$$

❖ Complex wave number in good conductors ($Q \ll 1$):

$$k = \frac{1-i}{\sqrt{2}} \sqrt{\mu_0 \sigma \omega}$$

❖ For copper, the good conductor condition $Q \ll 1$ is satisfied in the low-frequency regime ($\omega \ll 1/\tau_c$: radiowaves, microwaves).