UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 20

Low-frequency EM waves in good conductors

- Wave attenuation in good conductors
- Wave impedance in good conductors
- Reflection at vacuum/good conductor boundary
- Skin effect

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Previous lecture

- ❖ Ohm's law: $\vec{j} = \sigma \vec{E}$
- ❖ Static conductivity, applies to the low-frequency regime ($\omega \ll \gamma = 1/\tau_c$): $$\sigma = \frac{n_e e^2}{m_e} \tau_c$$
- ❖ Complex wave number in conductors, leading to attenuation of EM waves: $$k^2 = \frac{\omega^2}{c^2} - i\mu_0 \omega \sigma$$
- ❖ Ratio of displacement/conduction terms: $$Q = \frac{\omega^2/c^2}{\mu_0 \omega \sigma} = \frac{\omega \varepsilon_0}{\sigma}$$
- ❖ Complex wave number in good conductors ($Q \ll 1$):

$$k = \frac{1-i}{\sqrt{2}} \sqrt{\mu_0 \sigma \omega}$$

- ❖ For copper, the good conductor condition $Q \ll 1$ is satisfied in the low-frequency regime ($\omega \ll 1/\tau_c$: radiowaves, microwaves).

Wave attenuation

Plane wave in a good conductor ($k=k_r-ik_i$), in the **z** direction:

$$E_x = E_0 e^{i(\omega t - kz)} = E_0 e^{i(\omega t - k_r z)} e^{-k_i z}$$

Skin depth: distance corresponding to amplitude attenuation by a factor of **e**

$$\delta = \frac{1}{k_i} = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$$

For copper ($\sigma=6\times10^{7}\ \Omega^{-1}\text{m}^{-1}$; recall that $\mu_0 = 4\pi\times10^{-7}$ H/m), $\delta=10$ mm at $\nu=50$ Hz; and $\delta=10\ \mu$m at $\nu=50$ MHz.

No waves in ideal conductors: $\sigma\to\infty$, $k_i\to\infty$, $\delta\to0$, $\vec{E}=\vec{0}$.

Ratio of skin depth to wavelength in free space ($\lambda_0=2\pi/k_0=2\pi c/\omega$):

$$\frac{\delta}{\lambda} = \sqrt{\frac{2}{\mu_0\sigma\omega}}\cdot\frac{\omega}{2\pi c} = \sqrt{\frac{2\omega}{\mu_0\sigma}}\frac{\sqrt{\varepsilon_0\mu_0}}{2\pi} = \frac{1}{\sqrt{2}\pi}\sqrt{\frac{\omega\varepsilon_0}{\sigma}} = \frac{\sqrt{Q}}{\sqrt{2}\pi} \ll 1$$

Radiowaves and microwaves are rapidly attenuated in good conductors: RF shielding of sensitive equipment.

Wave impedance

Eq. (M3) for $\varepsilon = \mu = 1$: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

Using the shortcuts *(lecture 14)*, $-i\vec{k} \times \vec{E} = -i\omega\mu_0\vec{H}$

For a good conductor, $k = e^{-i\pi/4}\sqrt{\mu_0\sigma\omega}$ *(lecture 19)*

Therefore, the wave impedance is

$$Z = \frac{E}{H} = \frac{\mu_0\omega}{k} = \mu_0\omega \cdot e^{i\pi/4}\frac{1}{\sqrt{\mu_0\sigma\omega}} = e^{i\pi/4}\sqrt{\frac{\mu_0\omega}{\sigma}}$$

Comparing to the expression for the skin depth, $\delta = \sqrt{\frac{2}{\mu_0\sigma\omega}}$,

we obtain $Z = \frac{1+i}{\sqrt{2}} \cdot \frac{1}{\sigma}\sqrt{\mu_0\sigma\omega} = \frac{1+i}{\sigma\delta}$

Magnetic field *lags in phase by 45 degrees* wrt electric field.

EM field energy

Example: copper (σ=6×10^7 Ω^{-1}m^{-1}; μ_0=4π×10^{-7} H/m) at ν=10 GHz,

$$Z = \frac{i+1}{\sqrt{2}}\sqrt{\frac{\mu_0\omega}{\sigma}} = 0.026\cdot(i+1)\ \Omega$$

Ratio of electric and magnetic field amplitudes:

$$|Z| = \left|\frac{E_0}{H_0}\right| = \sqrt{\frac{\mu_0\omega}{\sigma}} = \frac{\mu_0}{\sqrt{\varepsilon_0\mu_0}}\sqrt{\frac{\omega\varepsilon_0}{\sigma}} = \mu_0 c\sqrt{Q} \ll \mu_0 c$$

Recall that the impedance of free space is Z_0 = E/H = μ_0c = 377 Ω, and the electric/magnetic energy ratio in free space is

$$\left(\frac{1}{2}\varepsilon_0 E^2\right) / \left(\frac{1}{2\mu_0}B^2\right) = \varepsilon_0\mu_0\cdot\frac{E^2}{B^2} = \varepsilon_0\mu_0 c^2 = 1$$

(lecture 14)

EM field energy in a good conductor is almost entirely magnetic.
Large conductivity σ leads to small E due to $\vec{j} = \sigma\vec{E}$.

Refractive index

Refractive index definition via the phase velocity of the wave:

$$v = \frac{\omega}{k} = \frac{c}{n}, \quad \text{therefore} \quad n = \frac{kc}{\omega}$$

Complex wave number is equivalent to *complex refractive index*. Similarly to dielectrics (*lecture 18*), this leads to wave attenuation.

Refractive index of good conductors ($\omega \ll \sigma/\varepsilon_0$) in the low-frequency regime ($\omega \ll \gamma = 1/\tau_c$):

Recall that $c = 1/\sqrt{\varepsilon_0 \mu_0}$

$$n = \frac{c}{\omega} \cdot \frac{1-i}{\sqrt{2}} \sqrt{\mu_0 \sigma \omega} = \frac{1-i}{\sqrt{2}} \cdot \sqrt{\frac{\sigma}{\varepsilon_0 \omega}} = \frac{1-i}{\sqrt{2Q}}$$

As for the wave number, equal real and imaginary parts:

$$n = n_r - i n_i \text{ , with } n_r = n_i = 1/\sqrt{2Q}$$

Considering that $Q \ll 1$, we conclude that $n_r \gg 1$, $n_i \gg 1$.

Reflection at conductor surface

Reflectance at a boundary, normal incidence (*lecture 17*):

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

This expression can be applied to conductors (with complex **n** values).
For a vacuum/good conductor boundary,

$$R = \left|\frac{n-1}{n+1}\right|^2 = \left|\frac{\frac{1-i}{\sqrt{2Q}} - 1}{\frac{1-i}{\sqrt{2Q}} + 1}\right|^2$$

Let's denote $x = n_r = 1/\sqrt{2Q} \gg 1$.

Keeping only the leading term in **1/x**,

$$R = \left|\frac{(1-i)x - 1}{(1-i)x + 1}\right|^2 = \frac{|-1 + x - ix|^2}{|1 + x - ix|^2} = \frac{(x-1)^2 + x^2}{(x+1)^2 + x^2}$$

$$\approx \frac{2x^2 - 2x}{2x^2 + 2x} = \frac{x-1}{x+1} = \frac{1 - 1/x}{1 + 1/x} \approx \left(1 - \frac{1}{x}\right)^2 \approx 1 - \frac{2}{x}$$

Reflection at conductor surface

Reflectance at normal incidence: $R = 1 - 2/n_r = 1 - 2\sqrt{2Q}$

Transmittance: $T = 1 - R = 2\sqrt{2Q} = 2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}} \ll 1$

Metals: near-perfect reflection of radiowaves and microwaves, (λ>1 mm), thanks to the large values of n_r and n_i.

Example: copper at a microwave frequency $\omega = 10^{10}$ s^{-1},

$$T = 2\sqrt{\frac{2 \cdot 8.9 \times 10^{-12}\ \mathrm{F/m} \cdot 10^{10}\ \mathrm{Hz}}{6 \times 10^{7}\ \Omega^{-1}\mathrm{m}^{-1}}} \approx 10^{-4}$$

The transmitted wave is dissipated as Joule heating.

Optical frequencies do not satisfy the condition $\omega \ll 1/\tau_c \sim 10^{13}$ s^{-1}.

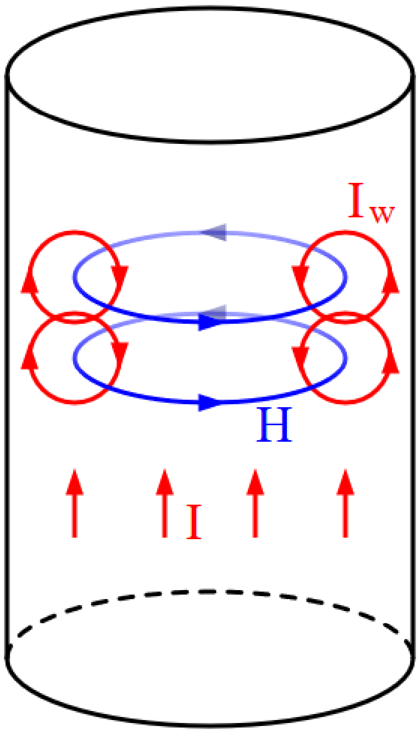
The classical theory is unable to describe reflection at optical frequencies ($\omega \sim 10^{15}$ s^{-1}); atomic transitions (E~1 eV) take place.

Skin effect

Alternating current in a good conductor, $Q = \frac{\omega\varepsilon_0}{\sigma} \ll 1$

The **E** field penetrates a distance δ into the surface, $\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$

Eddy currents arising from changing **B** field cancel the current in the centre of conductor and reinforce it in the skin.



Resistance of a cylindrical wire (length L, radius r):

at low frequency, $R_0 = \frac{1}{\sigma} \cdot \frac{L}{\pi r^2}$;

at higher frequency ω (such that δ≪r), the current is carried in a thin skin layer of a typical width δ;

$$R \approx \frac{1}{\sigma} \cdot \frac{L}{2\pi r \delta} = R_0 \cdot \frac{r}{2\delta} \gg R_0$$

Copper wire, r=1 mm, ν=ω/2π=50 MHz: $R/R_0 = r/(2\delta) = 50$.

For high-frequency currents in general, $R \sim \frac{1}{\sigma\delta} \sim \sqrt{\frac{\omega}{\sigma}}$

Summary

❖ Rapid attenuation of waves in good conductors, characterised by the skin depth:

$$\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$$

❖ Refractive index and wave impedance in good conductors:

$$Z = \frac{E}{H} = \frac{1+i}{\sigma\delta} \qquad |Z| \ll \mu_0 c \qquad n = \frac{1-i}{\sqrt{2Q}}$$

- ✓ Magnetic field lags in phase by 45 degrees.
- ✓ Field energy is almost entirely magnetic.
- ✓ Large real and imaginary parts of refractive index.

❖ Good conductors reflect low-frequency EM waves very well.

Transmittance (normal incidence): $T = 2\sqrt{2Q} \ll 1$

❖ Skin effect in good conductors: wire resistance for high-frequency currents

$$R \sim \frac{1}{\sigma\delta} \sim \sqrt{\frac{\omega}{\sigma}}$$