UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture 21
EM waves in plasma

- Plasma oscillations, plasma frequency
- Refractive index of plasma
- Dispersion relation in plasma
- Properties of EM waves in plasma

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Previous lecture

❖ Rapid attenuation of waves in good conductors, characterised by the skin depth:

$$\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$$

❖ Refractive index and wave impedance in good conductors:

$$Z = \frac{E}{H} = \frac{1+i}{\sigma\delta} \qquad |Z| \ll \mu_0 c \qquad n = \frac{1-i}{\sqrt{2Q}}$$

- ✓ Magnetic field lags in phase by 45 degrees.
- ✓ Field energy is almost entirely magnetic.
- ✓ Large real and imaginary parts of refractive index.

❖ Good conductors reflect low-frequency EM waves very well.

Transmittance (normal incidence): $T = 2\sqrt{2Q} \ll 1$

❖ Skin effect in good conductors:
wire resistance for high-frequency currents

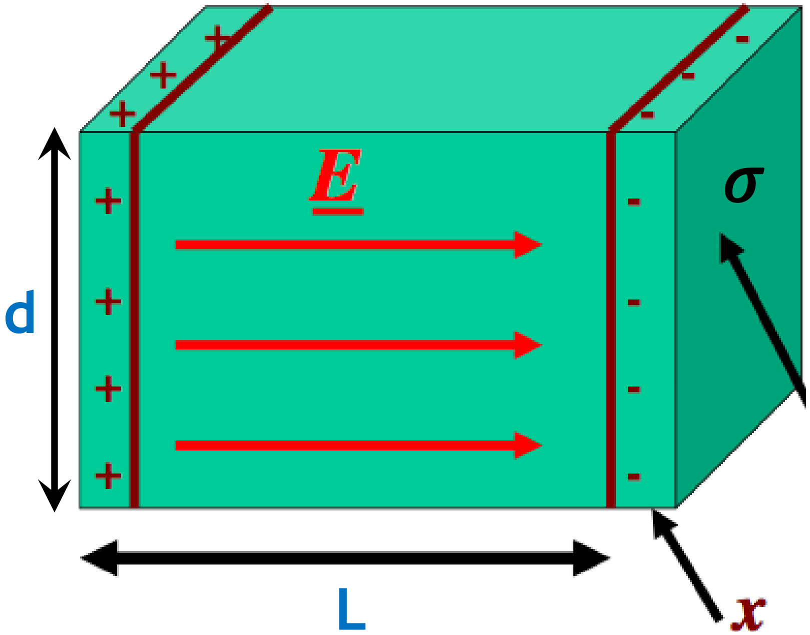
$$R \sim \frac{1}{\sigma\delta} \sim \sqrt{\frac{\omega}{\sigma}}$$

Plasma

Plasma: fully or partially ionised but electrically neutral gas of unbound positive and negative particles.

- Density of unbound charged particles must be sufficiently high: this leads to *collective effects*.
- The universe largely consists of plasma: stars, nebulae, interstellar and intergalactic media, …
- On Earth: magnetosphere, ionosphere, lightning.
- Practical applications: the tokamak.

Plasma oscillations



Consider all electrons displaced by a distance x≪L from their equilibrium positions wrt the positive ions.

Surface charge density for displacement x:

$$\sigma = \pm n_e e x$$

For d≫L, uniform electric field (by Gauss law): $E = \frac{n_e e x}{\varepsilon_0}$

Equation of motion of *each electron*: $m\ddot{x}(t) = -\frac{n_e e^2}{\varepsilon_0} x(t)$
(a harmonic oscillator)

A steady solution: $x(t) = x_0 e^{i\omega_P t}$, with $\omega_P = \sqrt{\frac{n_e e^2}{m_e \varepsilon_0}}$

The *plasma frequency* ω_P: the resonant frequency of *collective oscillations* of electrons about their equilibrium positions.
Oscillations of the (heavy) ions can often be neglected.

The Debye length

(not discussed in the lecture)

Potential energy of each electron, displaced by distance **x**:

$$U(x) = \int_0^x eE(x')dx' = \frac{1}{2}\cdot\left(\frac{n_e e^2}{\varepsilon_0}\right)x^2$$

Maximum displacement **x** possible: all thermal energy becomes potential energy,

$$kT \approx \frac{1}{2}\cdot\left(\frac{n_e e^2}{\varepsilon_0}\right)x^2$$

Displacement satisfying this condition is called the *Debye length* or *Debye radius*:

$$\lambda_D \approx \sqrt{\frac{\varepsilon_0 kT}{n_e e^2}}$$

It quantifies the *screening of electric changes* in plasma. Numerically, it ranges from **λ_D~0.1 mm** in a tokamak to **λ_D~100 km** in intergalactic medium.

Quantitative definition of ideal plasma: $n_e \lambda_D^3 \gg 1$

i.e. many electrons in a Debye sphere (leading to collective effects).

Free electrons in plasma

Consider an oscillating electric field, $\vec{E} = \vec{e}_x E_0 e^{i\omega t}$

Reminder: equation of motion of a *free electron in conductor* is

$$m\ddot{x} = \underbrace{-qE_0 e^{i\omega t}}_{\text{Electric force.}} \underbrace{- m\gamma\dot{x}}_{\text{Dissipative (damping) term}}$$ *(lecture 19)*

Electric force.
q>0: elementary charge.

Dissipative (damping) term:
collisions with lattice.

Plasma has much smaller density than conductors.
The damping constant γ (mean frequency of collisions) is negligible.
Therefore, motion of free electrons *in plasma*:

$$m\ddot{x} = -qE_0 e^{i\omega t}$$

This equation also applies to **conductors** in the **high-frequency regime** not considered so far ($\omega \gg \gamma = 1/\tau_c$, often valid for X-rays).

Steady solution: harmonic oscillation *in phase* with the **E**-field,

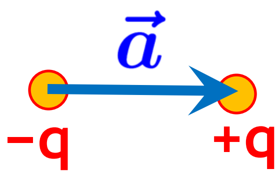
$$x(t) = \frac{q}{m\omega^2} E_0 e^{i\omega t} = \frac{q}{m\omega^2} E(t)$$

Wave propagation in plasma

Polarisation: induced dipole moment per unit of volume *(lecture 7)*

$$P(t) = -n_e e x(t) = -\frac{n_e e^2}{m\omega^2} \cdot E(t)$$

$$\vec{p} = q\vec{a}$$



(note the minus sign, unlike dielectrics in static regime)

The relative permittivity *(lecture 8)* is real:

$$\varepsilon = 1 + \chi_E = 1 + \frac{P}{\varepsilon_0 E} = 1 - \frac{n_e e^2}{\varepsilon_0 m\omega^2} = 1 - \frac{\omega_P^2}{\omega^2} < 1$$

Refractive index *(lecture 14)*: $n = \sqrt{\varepsilon} = \sqrt{1 - \omega_P^2/\omega^2}$

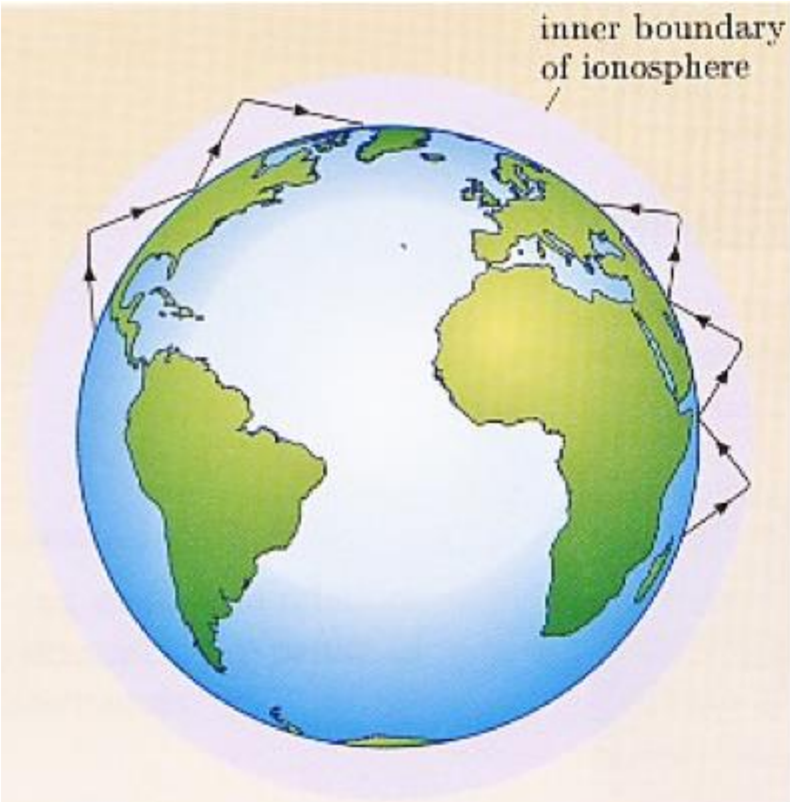
For $\omega > \omega_P$: real refractive index, $n(\omega) < 1$, plasma is transparent.
For the same reason, metals are transparent to X-rays.

For $\omega < \omega_P$: imaginary refractive index, $n = i\sqrt{\omega_P^2/\omega^2 - 1} = i n_i$,
EM waves do not propagate (evanescent fields).

Reflectance at vacuum/plasma boundary for $\omega < \omega_P$:
R=1 for any angle of incidence (Fresnel equations, *lecture 17*).

Example: normal incidence, $R = \left|\frac{1-n}{1+n}\right|^2 = \frac{1+n_i^2}{1+n_i^2} = 1$

Radio communication



Radiowaves are reflected by the *ionosphere* (upper layer of the atmosphere; **~100 km** above surface; plasma with electron density n_e~10^{12} m^{-3}), and by ground or water (which are good conductors).

This enables long-range radio communication.

1901: Marconi demonstrated transmission of **1 MHz** radiowaves across the Atlantic. No explanation was given why this was possible.

1902: Kennelly and Heaviside suggested the existence of a conducting layer in the upper atmosphere.

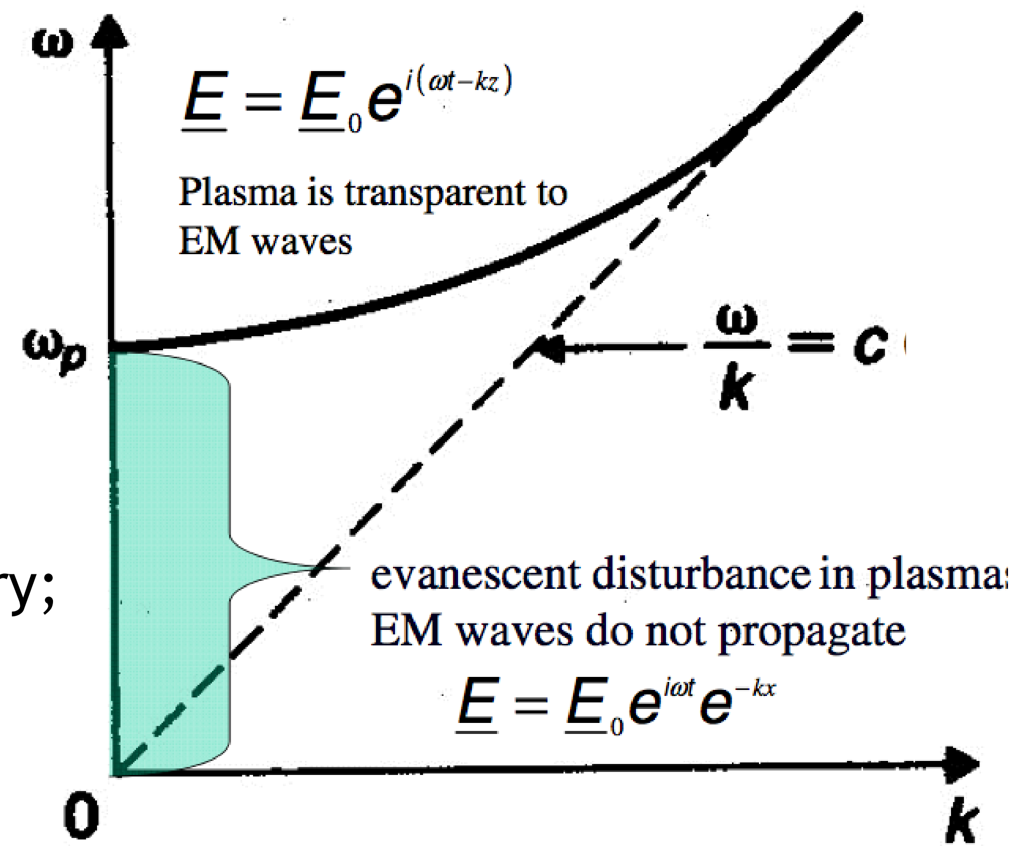
1924: Appleton confirmed the existence of the ionosphere experimentally (Nobel Prize **1947**).

Dispersion relation for plasma

Dispersion relation, i.e. the relation between the wave number **k** and the wave frequency **ω**:

$$k^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{\omega n}{c}\right)^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_P^2}{\omega^2}\right) = \frac{\omega^2}{c^2} - \frac{\omega_P^2}{c^2}$$

$$\omega(k) = \sqrt{\omega_P^2 + k^2c^2}$$



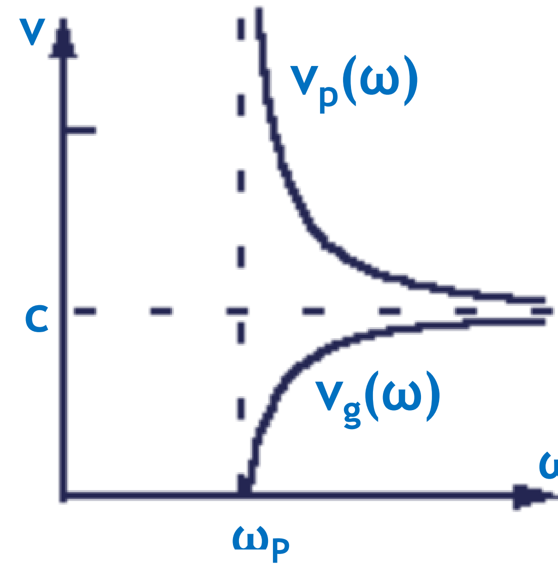
- For $\omega > \omega_P$, the wave number **k** is real; plasma is transparent for EM waves.
- For $\omega < \omega_P$, **k** is purely imaginary; EM waves do not propagate; (evanescent fields).

Speed of EM waves in plasma

Phase velocity of EM waves ($\omega > \omega_P$):

$$v_\mathrm{p} = \frac{\omega}{k} = \frac{c}{n} = \frac{c}{\sqrt{1-(\omega_P/\omega)^2}} > c$$

Note that $v_\mathrm{p} \to \infty$ for $\omega \to \omega_P$.



This is the speed of a point with a fixed phase.
Signal transmission requires wave modulation.
A *wave packet* travels with the *group velocity*:

$$v_\mathrm{g} = \frac{d\omega}{dk}$$

Differentiating the dispersion relation, we find

$$2kdk = \frac{2\omega d\omega}{c^2} \;; \quad \frac{w}{k} \cdot \frac{dw}{dk} = v_\mathrm{p} v_\mathrm{g} = c^2$$

Finally,

$$v_\mathrm{g} = \frac{d\omega}{dk} = \frac{c^2}{v_\mathrm{p}} = c\sqrt{1-(\omega_P/\omega)^2} < c$$

Properties of waves in plasma (1)

A plane wave propagating in the **z** direction:

$$E_x = E_0 e^{i(\omega t - kz)}$$
$$H_y = H_0 e^{i(\omega t - kz)}$$

Eq. **(M3)** for $\varepsilon = \mu = 1$: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

Using the shortcuts *(lecture 14)*, $-i\vec{k} \times \vec{E} = -i\omega\mu_0 \vec{H}$

Therefore

$$H_y = \frac{k}{\mu_0 \omega} E_x = \frac{1}{\mu_0 \omega}\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_P^2}{c^2}} E_x = \frac{1}{\mu_0 c}\sqrt{1 - \frac{\omega_P^2}{\omega^2}} E_x$$

Wave impedance:

Impedance of free space *(lecture 14)*

$$Z = \frac{E}{H} = \frac{\mu_0 c}{\sqrt{1 - (\omega_P/\omega)^2}} = \frac{Z_0}{\sqrt{1 - (\omega_P/\omega)^2}}$$

Properties of waves in plasma (2)

$$k^2 = \frac{\omega^2}{c^2} - \frac{\omega_P^2}{c^2} \qquad H_y = \frac{1}{\mu_0 c}\sqrt{1 - \frac{\omega_P^2}{\omega^2}}E_x$$

- High frequencies ($\omega > \omega_P$): E and H fields are in phase; impedance $Z=E/H$ and wave number k are real; no attenuation. For $\omega \gg \omega_P$, the wave in not affected by the presence of plasma: $Z \rightarrow Z_0$, free electrons do not respond to the oscillating E-field.
- For $\omega \rightarrow \omega_P$, we see that $H \rightarrow 0$ and $k \rightarrow 0$. Free electrons oscillate. However, no wave propagation and no transport of energy. Poynting vector: $N=EH \rightarrow 0$.
- Low frequencies ($\omega < \omega_P$): impedance $Z=E/H$ is purely imaginary, i.e. E and H are out of phase by 90 degrees. The wave number k is purely imaginary: evanescent field.
- Physics of EM waves in plasma becomes much more complex In the presence of external magnetic fields.

Summary

- Plasma oscillations with a frequency of $\omega_P = \sqrt{\dfrac{n_e e^2}{m_e \varepsilon_0}}$ is an example of a collective phenomenon.
- Refractive index of plasma: $n = \sqrt{\varepsilon} = \sqrt{1 - \omega_P^2/\omega^2}$.
- Dispersion relation in plasma: $k^2 = \dfrac{\omega^2}{c^2} - \dfrac{\omega_P^2}{c^2}$.
- Frequencies below ω_P: waves are attenuated in plasma, and fully reflected at vacuum/plasma boundary at normal incidence.
- Frequencies above ω_P: waves propagate in plasma without attenuation, with a phase velocity $v_p>c$.
- These considerations also apply to conductors in the high-frequency regime.