UNIVERSITY OF BIRMINGHAM

Electromagnetism 2
(spring semester 2026)

Lecture 23

Motion of charges in uniform fields

- ❖ Motion in uniform electric and magnetic fields
- ❖ Cyclotron frequency and radius
- ❖ Principles of mass spectrometry
- ❖ Thomson's parabola spectrometer
- ❖ Electrostatic lenses (non-uniform field)

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Motion in uniform E or B field

The Lorentz force (valid also in the relativistic case):

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Uniform static electric field: $\vec{F} = q\vec{E} = \text{const}$ (parabolic trajectory)

Uniform static magnetic field:

$$\vec{F} = q(\vec{v}_{||} + \vec{v}_{\perp}) \times \vec{B} = q\vec{v}_{\perp} \times \vec{B}$$

(1) Motion along the **B**-field: **F=0**, therefore uniform motion.

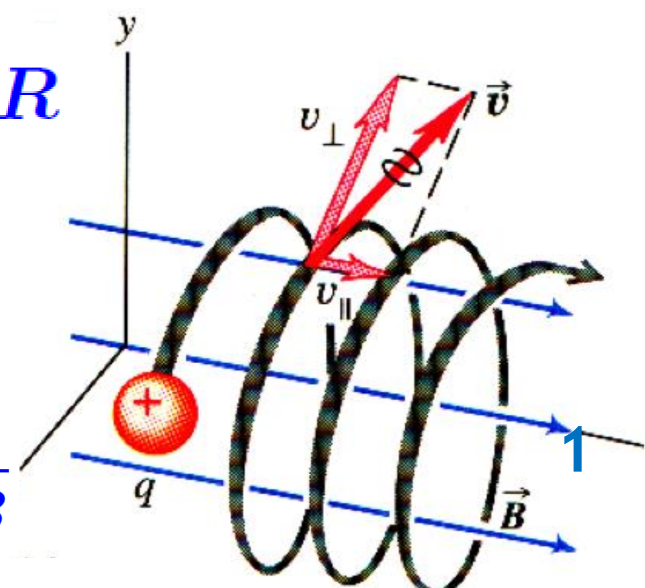
(2) In the plane perpendicular to **B**-field, **F⊥v**, circular motion with angular frequency ω.

Equation of motion: $F = qv_{\perp}B = m\omega^2 R$

$$q\omega RB = m\omega^2 R$$

In the vector form, $\vec{\omega} = -\frac{q\vec{B}}{m}$

Radius of curvature: $R = \frac{v_{\perp}}{\omega} = \frac{mv_{\perp}}{qB} = \frac{p_{\perp}}{qB}$



(1)+(2) constitute **helical motion**.

Uniform E and B fields (E⊥B)

Assume $E \ll cB$ (otherwise **B**-field can be neglected for **v≪c**).

Consider the non-trivial case $\vec{E} \perp \vec{B}$, $m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$

If **v'** is the velocity in a ref.frame moving with a velocity $\vec{v}_d$, then

$$m\frac{d\vec{v}'}{dt} = q(\vec{E} + \vec{v}' \times \vec{B} + \vec{v}_d \times \vec{B})$$

Let's choose $\vec{v}_d$ so that:

1) $\vec{E} + \vec{v}_d \times \vec{B} = 0$ (which is possible because $\vec{E} \perp \vec{B}$);

2) $\vec{v}_d \perp \vec{B}$ (which we are free to choose).

Then $\vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2} \ll c$ (a non-assessed problem: check this!)

Equation of motion in this reference frame: $m\frac{d\vec{v}'}{dt} = q\vec{v}' \times \vec{B}$

The effect of **E**-field is compensated by the choice of $\vec{v}_d$.

Therefore, **helical motion** in this reference frame.

Uniform E and B fields (general case)

Static uniform E and B fields (general **non-relativistic case**).
Consider the components of the **E**-field parallel and perpendicular to the **B**-field: $\vec{E} = \vec{E}_{||} + \vec{E}_{\perp}$

Superposition of three motions.

(1) *Motion with a constant acceleration* along the **B**-field

$$a_{||} = (q/m)E_{||}$$

(2) *Uniform rotation* about the **B**-field direction with a fixed angular frequency

$$\vec{\omega} = -\frac{q\vec{B}}{m}$$

(3) *Electric drift* with a velocity $\vec{v}_d = \frac{\vec{E}_{\perp} \times \vec{B}}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$,

independent of the mass and charge of the particle.

The components (1)+(2): helical motion with a variable pitch.

Cyclotron frequency and radius

A **relativistic charged particle** in a **uniform magnetic field**.

Consider velocity perpendicular to the **B**-field. Equation of motion:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{v}) = m\gamma\frac{d\vec{v}}{dt} + \underbrace{m\vec{v}\frac{d\gamma}{dt}}_{=0} = m\gamma\frac{d\vec{v}}{dt}$$

=0, no work is done on the charge by the magnetic field

Therefore $q\vec{v} \times \vec{B} = m\gamma\frac{d\vec{v}}{dt}$

Difference to the non-relativistic case: the Lorentz factor γ.

Therefore the angular frequency of rotation: the *cyclotron frequency*

(cf. Larmor frequency, lecture 9)

$$\omega_c = \frac{qB}{\gamma m}$$

Radius of curvature: the *cyclotron radius*

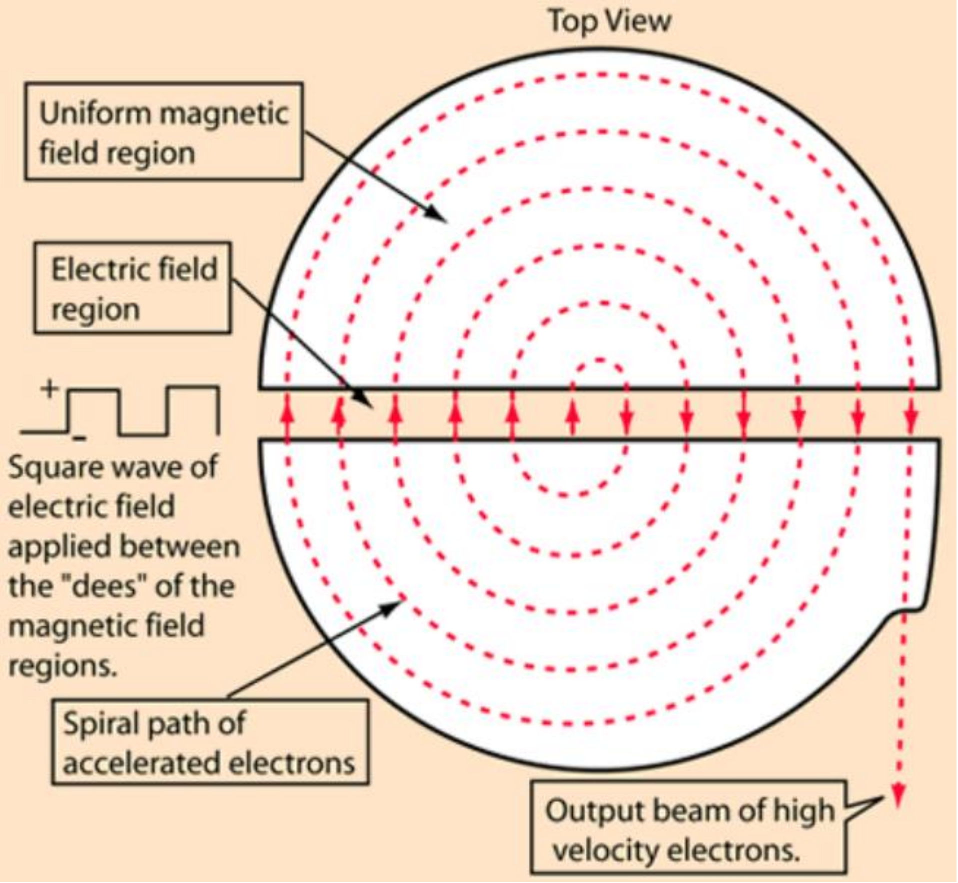
$$R = \frac{v}{\omega_c} = \frac{m\gamma v}{qB} = \frac{p}{qB}$$

The cyclotron

[not discussed in lecture]

The *cyclotron*: acceleration of non-relativistic particles ($\gamma \approx 1$) with electric field at each turn. Rotation at a fixed frequency ω, spiralling outwards ($R \sim p$).

In the relativistic case, rotation at a frequency $\omega_C \sim 1/\gamma$. The *synchrocyclotron*: electric field frequency is tuned correspondingly.



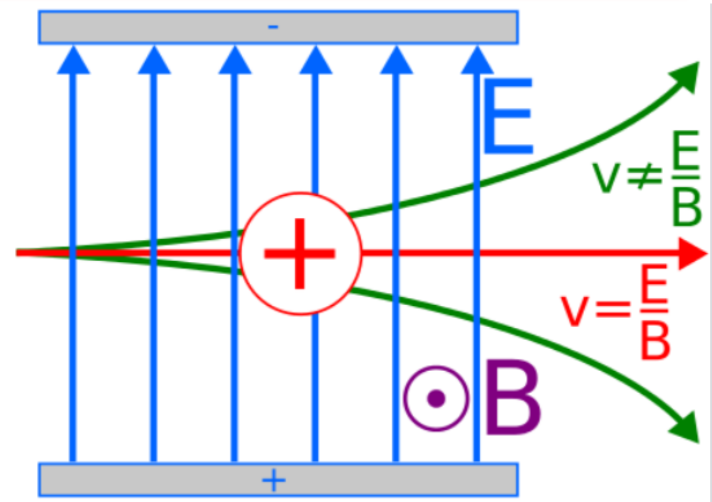
For elementary particles with **q=e** in magnetic field, $p_\perp = eBR$ (in SI units)

Dividing by **GeV/c**, obtain a useful relation for the bending radius:

$$p[\mathrm{GeV/c}] = 0.3 \cdot B[\mathrm{T}] \cdot R[\mathrm{m}]$$

Mass spectrometry (non-relativistic case)

Mass-spectrometry: measurement of the mass-to-charge (**m/q**) ratio.

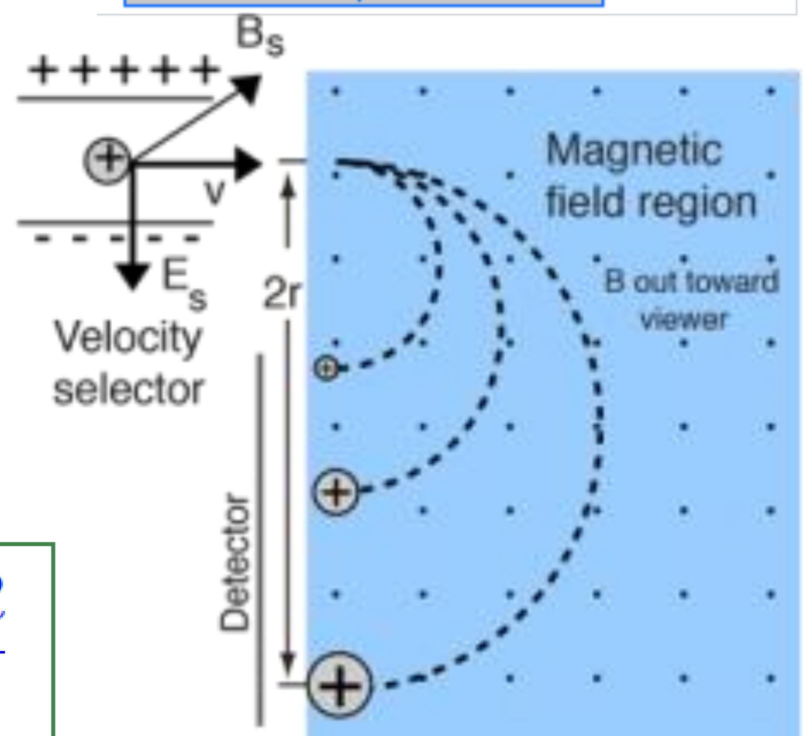


Velocity selector (*Wien filter*):
forces due to perpendicular electric and magnetic fields cancel for a specific speed, such that

$$qE = qvB \qquad v = E/B$$

For a beam of collimated and velocity-selected particles, the radius of the trajectory in a uniform **B**-field depends on **q/m** only.
Find **q/m** by measuring the radius:

$$R = \frac{mv}{qB} \quad \text{therefore} \quad m/q = \frac{BR}{v}$$



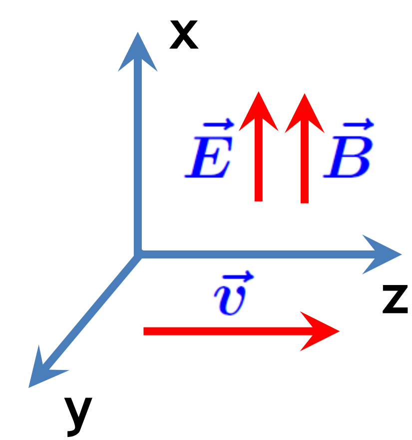
Thomson's parabola method (1)

A collimated beam of particles with a broad momentum spectrum.

E and B fields are not uniform, with $\vec{E} \parallel \vec{B}; \quad \vec{E} \perp \vec{v}$

Equations of motion:

$$m\frac{d^2x}{dt^2} = qE(z); \qquad m\frac{d^2y}{dt^2} = qvB(z)$$



Considering that (for $v_x, v_y \ll v_z$) $z = vt$,

$$\frac{d^2x}{dz^2} = \frac{q}{mv^2}E(z); \qquad \frac{d^2y}{dz^2} = \frac{q}{mv}B(z)$$

Double integration: $x(z) = A_E(z) \cdot \frac{q}{mv^2}; \quad y(z) = A_B(z) \cdot \frac{q}{mv}$

$A_E(z)$ and $A_B(z)$ are known constants, e.g. $A_E(z) = \int_0^z dz' \int_0^{z'} E(z'')dz''$

Thomson's parabola method (2)

In a fixed **z** plane,

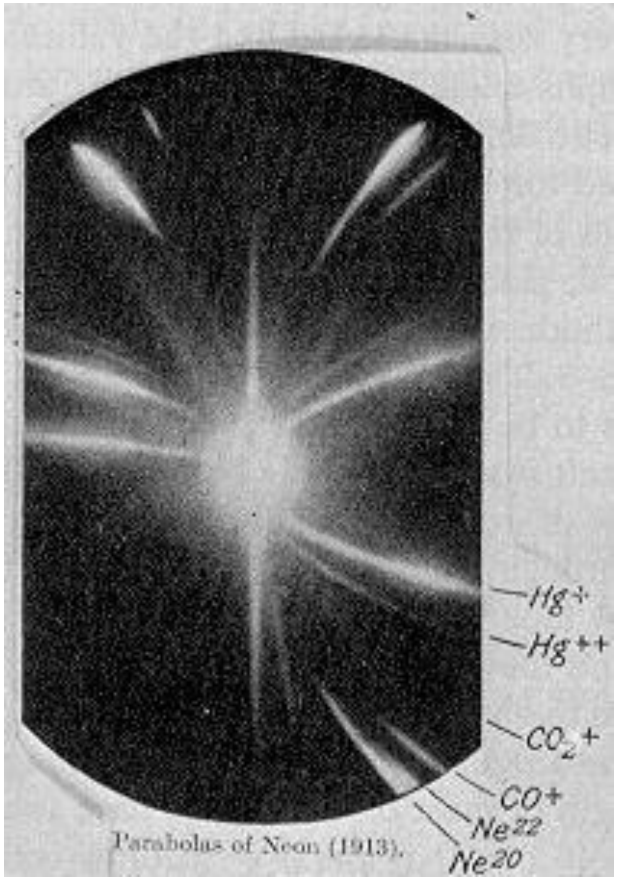
$$x = A_E \cdot \frac{q}{mv^2}; \quad y^2 = A_B^2 \cdot \frac{q^2}{m^2 v^2}$$

Therefore

$$x = (A_E / A_B^2) \cdot \frac{m}{q} y^2$$

Particles of fixed **m/q** lie on parabola;
its slope depends on **m/q**.

Thomson's parabola method:
discovery of stable ^{20}Ne and ^{22}Ne isotopes by J.J.Thomson in **1912**.



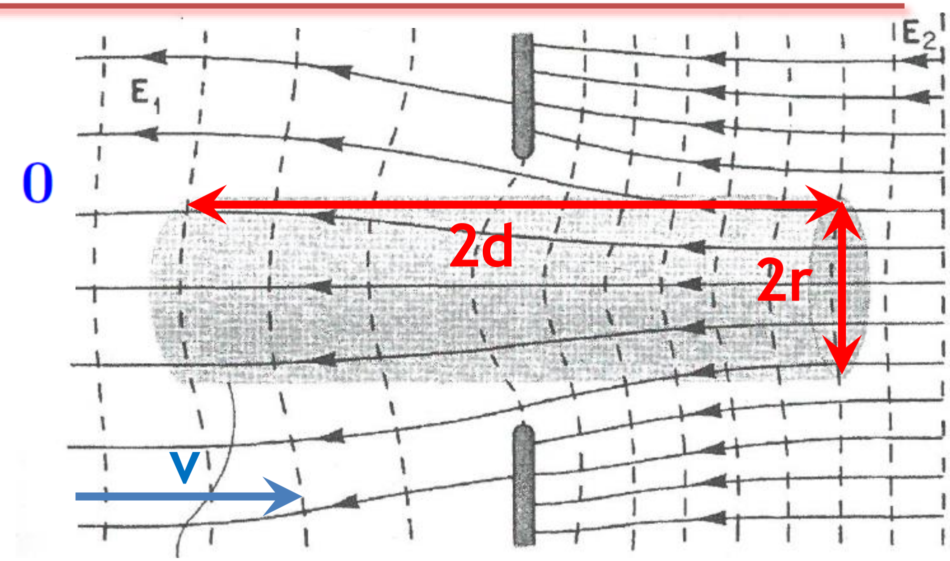
Electrostatic lenses

Gauss's law for the cylinder:

$$\pi r^2(E_1 - E_2) + 2\pi r \int_{-d}^{d} E_r dz = 0$$

Therefore

$$\int_{-d}^{d} E_r dz = \frac{1}{2} r(E_2 - E_1)$$



Radial momentum obtained by an electron traversing in any direction:

$$p_r = -\int_{-\infty}^{\infty} eE_r dt = -\int_{-d}^{d} eE_r \frac{dz}{v} = \frac{er}{2v}(E_1 - E_2)$$

(directed inwards: focusing)

Angle of deflection: $\alpha \approx \frac{|p_r|}{p} = \frac{|p_r|}{mv} = \frac{er}{2mv^2}(E_2 - E_1) = \frac{er}{4\mathcal{E}_e}(E_2 - E_1)$

Focal distance of the lens: $f = \frac{r}{\alpha} = \frac{4\mathcal{E}_e}{e(E_2 - E_1)}$

Example: for E_e=1 keV, and E_2−E_1=1 kV/cm, f=4 keV/1 keV=4 cm.

Summary

❖ Motion of charged particles in uniform **E**+**B** fields:
1) motion with a constant acceleration along the **B**-field;
2) uniform rotation about the **B**-field direction;
3) electrical drift with a velocity

$$\vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2}$$

❖ Cyclotron frequency and radius (including the relativistic case):

$$\omega_c = \frac{qB}{\gamma m} \quad \text{and} \quad R = \frac{p}{qB}$$

❖ In "particle physics" units, $p[\mathrm{GeV/c}] = 0.3 \cdot B[\mathrm{T}] \cdot R[\mathrm{m}]$

❖ Mass-spectrometry: measurement of the **m/q** ratio based on the trajectory in uniform **E**+**B** fields.