UNIVERSITY OF
BIRMINGHAM

Electromagnetism 2
(spring semester 2025)

Lecture A2 (non-examinable)
Transmission lines and waveguides

- Transmission lines
- Wave propagation in a parallel wire line
- Wave propagation in a rectangular waveguide

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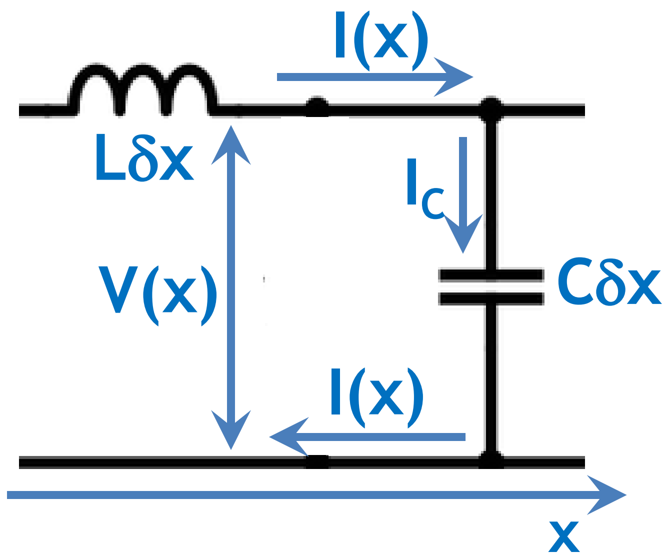
Transmission lines

Transmission line: a specialized structure designed for transmission of radiowaves (up to ω~1 GHz). Length of a line is comparable to, or larger than, the wavelength.

Examples: coaxial cable; twisted pair.



A lossless line: two conductors of zero resistance embedded in a perfect dielectric.

Need to account for the stray capacitance C and stray self-inductance L per unit length.

Faraday's law: the difference of voltage on two sides of a short section of length δx,

$$V(x+\delta x) - V(x) = \frac{\partial V}{\partial x}\delta x = -\frac{\partial \Phi}{\partial t} = -\frac{\partial (L\delta x \cdot I)}{\partial t} = -L\delta x\frac{\partial I}{\partial t}$$

Electric current:

$$I(x+\delta x) - I(x) = \frac{\partial I}{\partial x}\delta x = -\frac{\partial Q}{\partial t} = -\frac{\partial (C\delta x \cdot V)}{\partial t} = -C\delta x\frac{\partial V}{\partial t}$$

Transmission line: wave propagation

Equations for the voltage **V(x,t)** and current **I(x,t)**:

$$\begin{cases} \dfrac{\partial V}{\partial x} = -L\dfrac{\partial I}{\partial t} \\ \dfrac{\partial I}{\partial x} = -C\dfrac{\partial V}{\partial t} \end{cases}$$

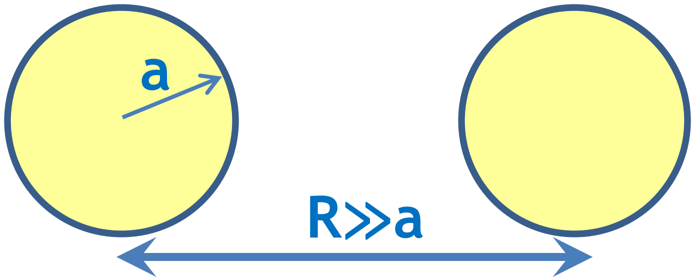
leading to

$$\begin{cases} -\dfrac{1}{L}\cdot\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial^2 I}{\partial x \partial t} \\ \dfrac{\partial^2 I}{\partial x \partial t} = -C\dfrac{\partial^2 V}{\partial t^2} \end{cases}$$

Finally, $\dfrac{\partial^2 V}{\partial x^2} = LC\dfrac{\partial^2 V}{\partial t^2} = \dfrac{1}{v^2}\dfrac{\partial^2 V}{\partial t^2}$ and similarly for **I**.

- This is the wave equation describing the propagation of EM disturbances with a speed of $v = 1/\sqrt{LC}$.
- The equation is identical to that for dielectric media *(lecture 13)*.
- In a lossless transmission line, waves of any frequency **ω** are transmitted without attenuation.

Parallel wire transmission line (1)



A pair of parallel cylindrical wires in a dielectric medium with relative permittivity ε and relative permeability μ.

$$\int_S \vec{D}d\vec{S} = Q$$

Gauss's law for the electric field at a distance x from the centre of a wire (σ: linear charge density; L: length of the wire considered):

$$\varepsilon_0 \varepsilon E(x) \cdot 2\pi x L = \sigma L$$, therefore $$E(x) = \frac{\sigma}{2\pi\varepsilon_0\varepsilon x}$$

Potential difference between wires with charge densities of +σ and −σ:

$$U = 2\int_a^R E(x)dx = \frac{\sigma}{\pi\varepsilon_0\varepsilon}\ln\left(\frac{R}{a}\right)$$

Capacitance per unit length: $C = \frac{\sigma}{U} = \frac{\pi\varepsilon_0\varepsilon}{\ln(R/a)}$

Parallel wire transmission line (2)

Magnetic field at a distance **x** from a wire from Ampere's law:

$$B(x) = \frac{\mu_0 \mu I}{2\pi x}$$

$$\oint_L \vec{H}\,d\vec{l} = I$$

Magnetic flux through section of length **d** between two wires with opposite currents of magnitude **I**:

$$\Phi = 2d \int_a^R B(x)\,dx = \frac{\mu_0 \mu I d}{\pi} \ln\left(\frac{R}{a}\right)$$

Self-inductance per unit length: $L = \frac{\Phi}{Id} = \frac{\mu_0 \mu}{\pi} \ln\left(\frac{R}{a}\right)$

Wave propagation speed $v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\varepsilon_0 \varepsilon \mu_0 \mu}} = \frac{c}{\sqrt{\varepsilon \mu}} = \frac{c}{n}$

is equal to the speed of light on the medium in which the line is embedded, for any frequency *(this is true for any lossless line)*.

Transmission lines are used for frequencies up to **~1 GHz**.
Coaxial cables are more efficient than parallel wire lines (no radiation of waves) but still limited by the skin effect.

Perfect conductor boundary

Waveguide: a structure designed to transmit microwaves (ω~10 GHz), a hollow metal pipe of constant cross-section.

Boundary conditions for oscillating EM fields near a a perfectly conducting surface (conductivity $\sigma \to \infty$).

1) **Electric field**: field amplitude falls off exponentially (*lecture 20*), the skin depth is $\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}} \to 0$

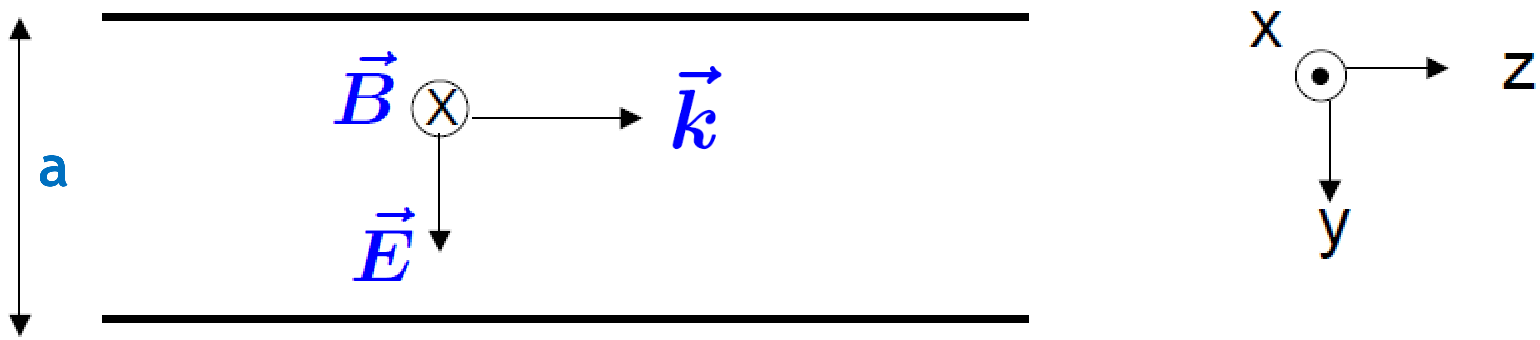
Therefore, no electric field inside perfect conductor (for the static case, see *lecture 3*).

The boundary condition $E_{1t} = E_{2t}$ leads to $\boxed{E_t = 0}$

2) **Magnetic field**: by Lenz's law, changing **B**-field induces currents acting to oppose a change. Perfect conductor expels oscillating magnetic fields.

The boundary condition $B_{1n} = B_{2n}$ leads to $\boxed{B_n = 0}$

TEM waves



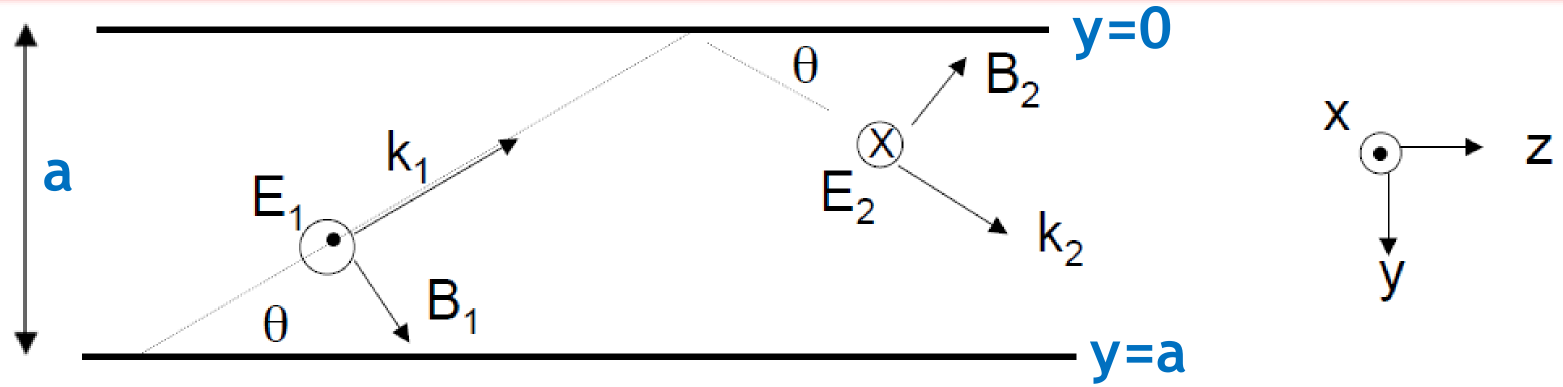
A linearly polarized wave propagating in free space along the z axis,

$$\vec{E} = \vec{e}_y E_0 e^{i(\omega t - kz)}; \qquad \vec{B} = -\vec{e}_x \frac{E_0}{c} e^{i(\omega t - kz)}$$

can also propagate between two infinite conducting plates (y=const), as it satisfies the boundary conditions at the walls $E_t = B_n = 0$.

This is a *TEM wave* ("transverse electric and magnetic"): both E and B fields are transverse to the k direction. Waves of this type also propagate in transmission lines. However they cannot propagate in waveguide (i.e. a hollow single conductor).

TE ("transverse electric") waves



Superposition of incident and reflected waves with wave vectors

$$\vec{k}_1 = -\vec{e}_y k \sin\theta + \vec{e}_z k \cos\theta; \quad \vec{k}_2 = \vec{e}_y k \sin\theta + \vec{e}_z k \cos\theta$$

Resulting wave propagates along **z**, with $E_z=0$, $B_z\neq 0$, hence a *TE wave*:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \vec{e}_x E_0 e^{i(\omega t - \vec{k}_1\vec{r})} - \vec{e}_x E_0 e^{i(\omega t - \vec{k}_2\vec{r})}$$

$$= \vec{e}_x E_0 e^{i\omega t}\left[e^{-i(-ky\sin\theta + kz\cos\theta)} - e^{-i(ky\sin\theta + kz\cos\theta)}\right]$$

$$= \vec{e}_x E_0 e^{i(\omega t - kz\cos\theta)}\left[e^{iky\sin\theta} - e^{-iky\sin\theta}\right]$$

$$= \vec{e}_x \cdot 2i\sin(ky\sin\theta) E_0 e^{i(\omega t - kz\cos\theta)}$$

A wave propagating in the **z** direction; standing wave in **y** direction.

Boundary conditions for E-field

The condition $E_x|_{y=0}=0$ is satisfied.
To satisfy the condition $E_x|_{y=a}=0$, require $\sin(ka\sin\theta)=0$

therefore $ka\sin\theta = n\pi, \quad n\in\mathbb{N}$

For a fixed wave number k, multiple solutions ($n=1,2,3,\ldots$) called *modes*, corresponding to *TE_{0n} waves*.

The *cut-off frequency*, or *critical frequency*, is determined by

$$1=\frac{n\pi}{k_{\min}a}=\frac{n\pi\lambda_{\max}}{2\pi a}=\frac{n\lambda_{\max}}{2a}$$

The smallest wavelength which can propagate in a rectangular waveguide (for the *TE_{01} wave*, $n=1$):

$$\lambda_{\max}=2a;\quad k_{\min}=\frac{2\pi}{\lambda_{\max}}=\frac{\pi}{a};\quad \omega_{\min}=\frac{2\pi c}{\lambda_{\max}}=\frac{\pi c}{a}$$

TM waves with the magnetic field perpendicular to the wave vector also exist, but not discussed here.

Boundary conditions for B-field

Eq. (M3): $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega\vec{B}$

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \vec{e}_y \cdot \frac{\partial E_x}{\partial z} - \vec{e}_z \cdot \frac{\partial E_x}{\partial y}$$

Therefore $B_x = 0; \quad -i\omega B_y = \frac{\partial E_x}{\partial z}; \quad -i\omega B_z = -\frac{\partial E_x}{\partial y}$

Finally, $B_x = 0; \quad B_y = (i/\omega)\frac{\partial E_x}{\partial z}; \quad B_z = -(i/\omega)\frac{\partial E_x}{\partial y}$

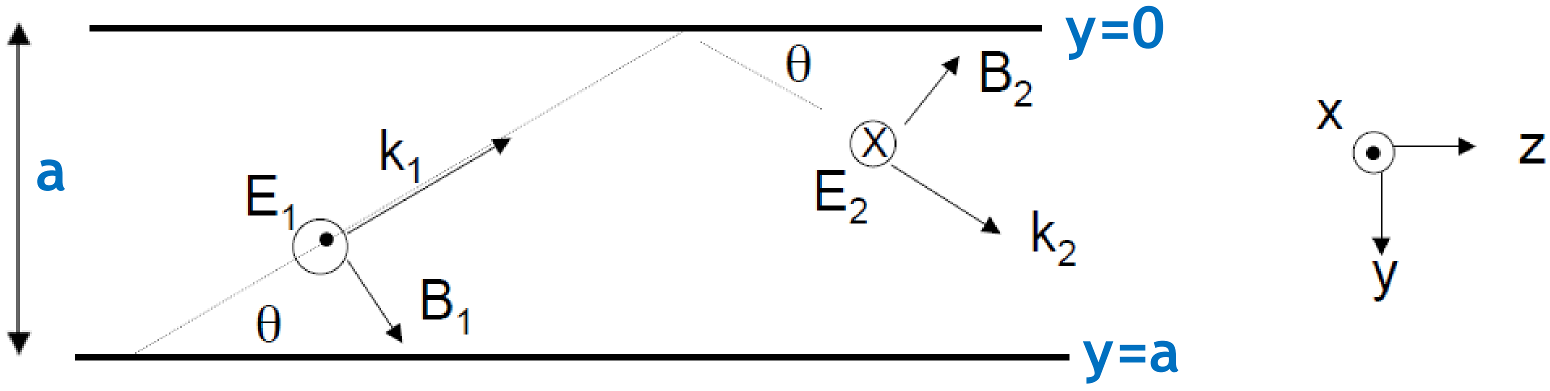
$$E_x = 2i\sin(ky\sin\theta)E_0 e^{i(\omega t - kz\cos\theta)}$$

$$B_y = \frac{2ik\cos\theta}{\omega}E_0\sin(ky\sin\theta)e^{i(\omega t - kz\cos\theta)}$$

$$B_z = \frac{2k\sin\theta}{\omega}E_0\cos(ky\sin\theta)e^{i(\omega t - kz\cos\theta)}$$

The condition $\sin(ka\sin\theta) = 0$ leads to $B_y=0$, i.e. $B_n=0$, at the boundaries $y=0$ and $y=a$.

Rectangular waveguide



Boundary conditions $E_t=B_n=0$ are satisfied in the planes $y=0$ and $y=a$.

Let's add two conducting walls parallel to the yz plane (i.e. x=const), at any positions x_1, x_2.

Considering that E is parallel to the x axis, and $B_x=0$, the boundary conditions on these walls are also satisfied:

$$E_t = 0 \qquad B_n = 0$$

Boundary conditions are satisfied on each of the 4 conducting walls. Therefore the TE waves can propagate in a rectangular waveguide.

TE_{01} wave: n=1, i.e. $ka\sin\theta = \pi$

Introduce the notation $k_y = k\sin\theta = \dfrac{\pi}{a};\quad k_z = k\cos\theta$

(k_y=const is determined by the boundary conditions at plane y=a).

Up to a constant phase, $E_x = E_0 \sin k_y y \cdot e^{i(\omega t - k_z z)}$

Considering $E_y=E_z=0$, the wave equation $\nabla^2 \vec{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \vec{E}}{\partial t^2}$ becomes

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

Therefore $-k_y^2 E_x - k_z^2 E_x = -\dfrac{\omega^2}{c^2} E_x$

The dispersion relation is $\boxed{k_y^2 + k_z^2 = \dfrac{\omega^2}{c^2}}$

The wave number of the propagating wave is $k_z = \sqrt{\dfrac{\omega^2}{c^2} - \dfrac{\pi^2}{a^2}}$

Phase and group velocities

The phase velocity is

$$v_{\rm p} = \frac{\omega}{k_z} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}}} = \frac{c}{\sqrt{1 - \left(\frac{\pi c}{\omega a}\right)^2}} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{\rm min}}{\omega}\right)^2}} > c$$

From the dispersion relation $k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$,

we obtain: $2k_z dk_z = \frac{2\omega d\omega}{c^2}$, therefore $\frac{\omega}{k_z} \cdot \frac{d\omega}{dk_z} = c^2$

The group velocity is $v_{\rm g} = \frac{d\omega}{dk_z} = \frac{c^2}{\omega_{\rm p}} = c\sqrt{1 - \left(\frac{\omega_{\rm min}}{\omega}\right)^2} < c$

Signal propagation slows down as the cut-off frequency is approached.

Note that $v_{\rm p} v_{\rm g} = c^2$, similarly to waves in plasma.

Summary

- Transmission lines: specialized cables designed to transmit radio frequency EM waves (up to **~1 GHz**).
  Waves of any frequency propagate at the same speed: the speed of light in the dielectric used.

- Waveguides: hollow metal pipes of constant cross-section used for transmission of microwaves (**~10 GHz**).

- Rectangular waveguides: TE waves of frequencies above

$$\omega_{\min} = \frac{2\pi c}{\lambda_{\max}} = \frac{\pi c}{a} \quad \text{i.e.} \quad \lambda_{\max} = 2a$$

  where **a** is the largest of the two transverse dimensions, can be transmitted. The dispersion relation:

$$k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$