Y4 Current Topics in Particle Physics - Fourier transform of Yukawa potential

∇^2 in Spherical Polar Co-ordinates

The formula for $\nabla^2 u$ in spherical polar co-ordinates is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

You have seen this before!

Fourier Transform of the Yukawa Potential

$$\begin{aligned} U(R) &= \frac{g}{4\pi r}e^{-\frac{r}{R}} \\ &= \frac{g}{4\pi r}e^{-mr} \end{aligned}$$

We want to evaluate

$$f(q) = g_0\int U(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r}$$

Now, in spherical polar co-ordinates for $\mathbf{r}$,

$$\mathbf{q}\cdot\mathbf{r} = qr\cos\theta$$

and

$$d^3\mathbf{r} = r^2\sin\theta d\theta d\phi dr$$

The integral over ϕ is easy, and we can substitute $\mu = \cos\theta$

$$\begin{aligned} f(q) &= g_0\int U(r)e^{iqr\cos\theta}r^2\sin\theta d\theta d\phi dr \\ &= 2\pi g_0\int U(r)e^{iqr\mu}r^2 d\mu dr \\ &= 2\pi g_0\int_0^\infty U(r)r^2\int_{-1}^{1}e^{iqr\mu}d\mu dr \\ &= 2\pi g_0\int_0^\infty U(r)r^2\left[\frac{e^{iqr\mu}}{iqr}\right]_{-1}^{1}dr \\ &= 2\pi g_0\int_0^\infty U(r)r^2\left(\frac{e^{iqr}}{iqr}-\frac{e^{-iqr}}{iqr}\right)dr \\ &= \frac{2\pi g_0}{iq}\int_0^\infty U(r)r\left(e^{iqr}-e^{-iqr}\right)dr \end{aligned}$$

Now substitute in $U(R) = \frac{g}{4\pi r} e^{-mr}$:

$$
\begin{aligned}
f(q) &= \frac{2\pi g_0}{iq} \int_0^\infty \frac{g}{4\pi r} e^{-mr} r \left(e^{iqr} - e^{-iqr} \right) dr \\
&= \frac{g_0 g}{2iq} \int_0^\infty e^{-mr} \left(e^{iqr} - e^{-iqr} \right) dr \\
&= \frac{g_0 g}{2iq} \int_0^\infty \left(e^{-r(m-iq)} - e^{-r(m+iq)} \right) dr \\
&= \frac{g_0 g}{2iq} \left[\frac{e^{-r(m-iq)}}{-(m-iq)} - \frac{e^{-r(m+iq)}}{-(m+iq)} \right]_0^\infty \\
&= \frac{g_0 g}{2iq} \left(\frac{-1}{-(m-iq)} - \frac{-1}{-(m+iq)} \right) \\
&= \frac{g_0 g}{2iq} \left(\frac{1}{(m-iq)} - \frac{1}{(m+iq)} \right) \\
&= \frac{g_0 g}{2iq} \left(\frac{m+iq-(m-iq)}{(m-iq)(m+iq)} \right) \\
&= \frac{g_0 g}{2iq} \left(\frac{2iq}{m^2+q^2} \right) \\
&= \frac{g_0 g}{(m^2+q^2)}
\end{aligned}
$$