Reconstruction Methods for HERA Kinematics

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1 Introduction

This document provides a guide to the different reconstruction methods that can be used to obtain the kinematic variables x, Q^2 and y in the deep-inelastic scattering process $ep \rightarrow eX$ (figure 1), when both the final state electron and the hadrons are reconstructed. Throughout, the following notation is adopted:

$$e = \text{4 vector of incoming electron.} \quad (1)$$
$$e' = \text{4 vector of scattered electron.} \quad (2)$$
$$p = \text{4 vector of incoming proton.} \quad (3)$$
$$q = \text{4 vector of exchange photon, W or Z.} \quad (4)$$
$$X = \text{4 vector of hadronic final state (everything except the scattered electron).} \quad (5)$$

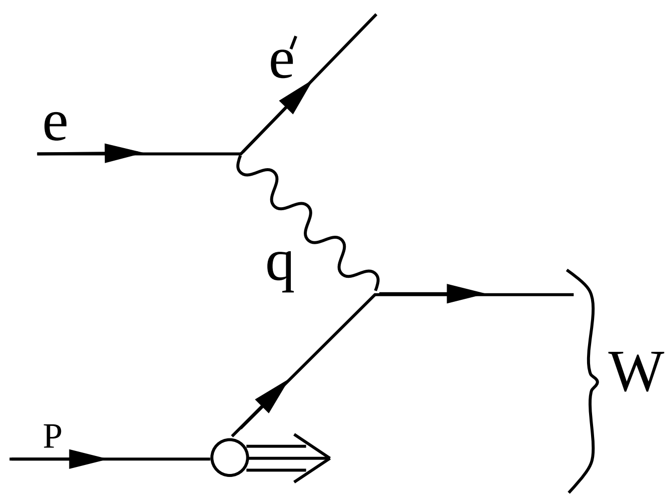

Figure 1: Diagram of the generic process $ep \rightarrow eX$. In the text, the four-vectors for each of the particles is as marked in the figure, except that the final state system (labelled W in the figure) has its four-vector denoted X.

The following are the formal definitions of the kinematic variables that we are trying to obtain. Look in any textbook for their physical interpretations and why they are interesting:

$$Q^2 = -q^2 \quad (6)$$

$$
\begin{aligned}
x &= \frac{-q^2}{2p \cdot q} & (7) \\
y &= \frac{p \cdot q}{p \cdot e} & (8)
\end{aligned}
$$

2 Relationships between the Kinematic Variables

For a two body $\rightarrow$ two body problem such as that under study, there are only two independent Lorentz scalars in addition to the centre of mass energy.

In general, we always reconstruct Q^2 and y directly and obtain x indirectly as follows:- If we also define the square of the total ep centre of mass energy as

$$
s = (e + p)^2 \tag{9}
$$

then we have

$$
\begin{aligned}
s &= m_e^2 + m_p^2 + 2e \cdot p & (10) \\
&\simeq 2e \cdot p & (11)
\end{aligned}
$$

where the masses m_e and m_p of the electron and proton are neglected. Now by comparing with equations 6 - 8, you should be able to convince yourself that

$$
Q^2 \simeq sxy. \tag{12}
$$

Since we always sit at fixed centre of mass energy $\sqrt{s}$, it is always possible to obtain x from equation 12 if we have reconstructed Q^2 and y.

As an aside, you may also come across the centre of mass energy of the hadronic part of the final state W, where

$$
W^2 = (q + p)^2. \tag{13}
$$

Equivalently, W is also the $\gamma^* p$ centre of mass energy. If we multiply this out, we get

$$
\begin{aligned}
W^2 &= -Q^2 + m_p^2 + 2q \cdot p & (14) \\
&\simeq -Q^2 + 2q \cdot p & (15)
\end{aligned}
$$

where the proton mass is again neglected. By comparing with equations 6 - 8, you can see that

$$
W^2 = Q^2 \left(\frac{1}{x} - 1 \right) \tag{16}
$$

3 4-vector Formalism

Since many different notations are used for 4-vectors, it is worth defining our terms before we start. In what follows, the components of a 4-vector are given as

$$(E,\ ip_x,\ ip_y,\ ip_z) \tag{17}$$

such that when 2 four-vectors are contracted, we can just multiply each of the terms in turn and add them. For example, to multiply a 4-vector by itself, we have

$$\begin{aligned}(E,\ ip_x,\ ip_y,\ ip_z)^2 &= E^2 - p_x^2 - p_y^2 - p_z^2 && (18)\\ &= m^2 && (19)\end{aligned}$$

where m^2 is the rest mass of the particle.

For the 4-vectors under consideration (equations 1 - 5), we thus have

$$\begin{aligned}e &= (E_e,\ 0,\ 0,\ -iE_e) && (20)\\ e' &= (E'_e,\ iE'_e \sin\theta,\ 0,\ iE'_e \cos\theta) && (21)\\ p &= (E_p,\ 0,\ 0,\ iE_p) && (22)\\ q &= e - e' && (23)\\ &= (E_e - E'_e,\ -iE'_e \sin\theta,\ 0,\ -i[E_e + E'_e \cos\theta]) && (24)\\ X &= (E_h,\ ip_t^h,\ 0,\ ip_z^h) && (25)\end{aligned}$$

where again, the proton and electron masses are neglected, the positive z direction is that of the incoming proton beam and for simplicity, we have defined the scattering to take place in the x/z plane. $E_e = 27.5$ GeV is the electron beam energy. $E_p = 820$ GeV is the proton beam energy. E'_e and θ are the energy and polar angle of the scattered electron respectively. E_h, p_t^h and p_z^h are the energy, transverse momentum and longitudinal momentum of the final state hadrons.

4 The 'electron' Reconstruction Method

Probably the most well known way of obtaining the kinematic variables is by using measurements of the scattered electron only. The derivation of the 'electron kinematics' is sketched out below.

From equations 20 - 25, we have

$$\begin{aligned}q^2 &= (E_e - E'_e)^2 - E'^2_e \sin^2\theta - (E_e + E'_e \cos\theta)^2 && (26)\\ &= -2E_e E'_e (1 + \cos\theta) && (27)\end{aligned}$$

such that (from equations 6 - 8), Q^2 from the electron method is

$$Q_e^2 = 2E_e E'_e (1 + \cos\theta)\ . \tag{28}$$

For y_e, we need to work out $p \cdot q / p \cdot e$. From equations 20 - 25, we have

$$\begin{aligned}
p \cdot q &= E_p(E_e - E'_e) + E_p(E_e + E'_e \cos\theta) && (29)\\
&= E_p[2E_e - E'_e(1-\cos\theta)] && (30)\\
p \cdot e &= 2E_pE_e && (31)\\
y &= \frac{p \cdot q}{p \cdot e} && (32)\\
&= \frac{2E_e - E'_e(1-\cos\theta)}{2E_e} && (33)\\
&= 1 - \frac{E'_e(1-\cos\theta)}{2E_e} && (34)
\end{aligned}$$

and using one of those instantly forgettable trig formulae:

$$y_e = 1 - \frac{E'_e}{E_e}\sin^2\frac{\theta}{2} \tag{35}$$

5 The 'hadron' Reconstruction Method

It is also possible to reconstruct the kinematic variables using information from the final state hadrons alone. In charged current interactions $ep \to \nu X$, this is the only method available.

To obtain y, first note that in terms of 4-vectors, we have $X = q + p$ and hence $q = X - p$, such that we can write

$$\begin{aligned}
y &= \frac{(X-p)\cdot p}{e \cdot p} && (36)\\
&= \frac{X \cdot p}{e \cdot p} - \frac{m_p^2}{e \cdot p} && (37)\\
&\simeq \frac{X \cdot p}{e \cdot p} && (38)
\end{aligned}$$

where the proton mass has been neglected again. Referring to equations 20 - 25, we then have

$$\begin{aligned}
X \cdot p &= E_pE_h - E_pp_z^h && (39)\\
&= E_p\Sigma && (40)
\end{aligned}$$

where the notation $\Sigma = E_h - p_z^h$ will be used extensively below. Given that

$$e \cdot p = 2E_eE_p \tag{41}$$

we thus end up with

$$\begin{aligned}
y &= \frac{E_p\Sigma}{2E_eE_p} && (42)\\
y_h &= \frac{\Sigma}{2E_e} && (43)
\end{aligned}$$

To obtain Q^2 using the hadrons only, we first need to perversely go back to the electron method for a moment. Consider the transverse momentum of the scattered electron:

$$p_t^{e'} = E'_e \sin\theta \tag{44}$$

such that

$$\begin{aligned} p_t^{e'2} &= E_e'^2 \sin^2\theta &(45)\\ &= E_e'^2(1-\cos^2\theta) &(46)\\ &= E_e'^2(1+\cos\theta)(1-\cos\theta) &(47)\end{aligned}$$

Now look back at equation 28 for Q^2 using the electron method and notice that

$$\begin{aligned} \frac{p_t^{e'2}}{Q^2} &= \frac{E_e'^2(1+\cos\theta)(1-\cos\theta)}{2E_eE'_e(1+\cos\theta)} &(48)\\ &= \frac{E'_e(1-\cos\theta)}{2E_e} &(49)\end{aligned}$$

and looking back at equation 34,

$$\frac{p_t^{e'2}}{Q^2} = 1-y \tag{50}$$

Now we know that the total transverse momentum of the final state must be zero if we are to conserve momentum. This implies that

$$p_t^{e'} = -p_t^h \tag{51}$$

such that we can replace equation 50 by

$$\frac{p_t^{h2}}{Q^2} = 1-y \tag{52}$$

and by rearranging, we finally end up with an expression for Q^2 based on the hadrons only:

$$Q_h^2 = \frac{p_t^{h2}}{1-y_h} \tag{53}$$

6 The 'sigma' Reconstruction Method

It turns out to be possible to get an improved resolution in many cases by mixing information from the electron and from the hadrons. In particular, the 'sigma' method is relatively insensitive to initial state radiation from the electron beam.

Look back at equation 43. We are going to replace the electron energy on the bottom line by final state variables using conservation laws and a sneaky trick. In fact, we are going to conserve the quantity $E - p_z$ through the interaction. For the electron, we have

$$E_e - p_z^e = 2E_e \tag{54}$$

since the incoming electron moves in the $-z$ direction and the electron mass is negligible. Since the proton beam goes in the $+z$ direction, we have

$$E_p - p_z^p = 0 \tag{55}$$

so that in total before the interaction, $E - p_z = 2E_e$. After the interaction, we have

$$(E - p_z)_{\rm tot} = \Sigma + E'_e(1 - \cos\theta) \tag{56}$$

and thus using the conservation laws:

$$2E_e = \Sigma + E'_e(1 - \cos\theta). \tag{57}$$

We now simply replace the bottom line of equation 43 and end up with

$$y_\Sigma = \frac{\Sigma}{\Sigma + E'_e(1 - \cos\theta)}\ . \tag{58}$$

In the sigma method, Q^2 again comes from the identity 50

$$Q^2 = \frac{p_t^{e'2}}{1 - y_\Sigma} \tag{59}$$

$$Q^2_\Sigma = \frac{E_e'^2 \sin^2\theta}{1 - y_\Sigma} \tag{60}$$

7 The 'e-sigma' Reconstruction Method

It gets even more complicated, but at least this one is simple! It turns out that you can often do even better by mixing the sigma and electron methods. In the e-sigma method, we just take

$$Q^2_{e\Sigma} = Q^2_e \tag{61}$$

$$x_{e\Sigma} = x_\Sigma \tag{62}$$

where equation 28 is used for Q^2 and x comes from equations 60 and 58 after using equation 12 to obtain x_Σ.

8 The 'double angle' Reconstruction Method

Imagine that you have a really dodgy calorimeter that has a very poor energy resolution - this is the case for example in the ZEUS experiment [1]. The double angle method allows you to reconstruct the kinematic variables without ever measuring an energy. All you

[1] A subtle dig at the opposition has to appear somewhere in every H1 document! This one was not so subtle!

need is the angle of the scattered electron (θ_e) and the overall angle of the final state hadrons (θ_h). It is also a very useful method if the hadronic final state is simple and well measured, for example in the process $ep \to e\rho^0 p \to e\pi^+\pi^- p$.

The maths is a little tricky: start by defining

$$\alpha_e = \tan\frac{\theta_e}{2} \tag{63}$$

$$\alpha_h = \tan\frac{\theta_h}{2} \tag{64}$$

then let's prove another identity, which is true both for the scattered electron and for the final state hadrons. We have the following with either e' or h subscripts on all variables:

$$E - p_z = E(1-\cos\theta) \tag{65}$$

$$p_t = E\sin\theta \tag{66}$$

such that

$$\frac{E-p_z}{p_t} = \frac{(1-\cos\theta)}{\sin\theta} \tag{67}$$

and using some more of those trig identities:

$$\frac{E-p_z}{p_t} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \tag{68}$$

$$= \tan\frac{\theta}{2} \tag{69}$$

so that we can write

$$\frac{E'_e - p_z^{e'}}{p_t^{e'}} = \alpha_e \tag{70}$$

$$\frac{E_h - p_z^h}{p_t^h} = \alpha_h \tag{71}$$

Now look back at equation 58 for y in the sigma method. It can be written as

$$y = \frac{E_h - p_z^h}{(E_h - p_z^h) + (E'_e - p_z^{e'})} \tag{72}$$

Dividing through by p_t and remembering that $p_t^h = p_t^{e'}$,

$$y = \frac{\frac{E_h - p_z^h}{p_t^h}}{\frac{E_h - p_z^h}{p_t^h} + \frac{E'_e - p_z^{e'}}{p_t^{e'}}} \tag{73}$$

which, through equations 70 - 71 gives

$$y_{\mathrm{DA}} = \frac{\alpha_h}{\alpha_h + \alpha_e}. \tag{74}$$

To obtain Q^2 in the double angle method, start by summing equations 70 - 71:

$$\alpha_e + \alpha_h = \frac{E_h - p_z^h}{p_t^h} + \frac{E_e' - p_z^{e'}}{p_t^{e'}} \quad (75)$$

$$= \frac{E_h - p_z^h + E_e' - p_z^{e'}}{p_t} \quad (76)$$

where the second line follows from the overall event p_t balance. Now the numerator of the right hand side of equation 76 is the total $E - p_z$ of the final state. We already showed in section 6 that by conseving $E - p_z$ through the interaction, this is equal to $2E_e$. Thus equation 76 can be written as

$$p_t = \frac{2E_e}{\alpha_e + \alpha_h} \quad (77)$$

The substituting for p_t in equation 50, we have

$$Q^2 = \frac{4E_e^2}{(1-y)(\alpha_e + \alpha_h)^2} \ . \quad (78)$$

Finally, substituting for y from equation 74, we get

$$Q^2 = \frac{4E_e^2}{(1 - \frac{\alpha_h}{\alpha_h + \alpha_e})(\alpha_e + \alpha_h)^2} \ . \quad (79)$$

and ultimately,

$$Q^2 = \frac{4E_e^2}{\alpha_e(\alpha_e + \alpha_h)} \ . \quad (80)$$