DEEP INELASTIC SCATTERING

What is it?

- Studies started at SLAC in the 1960s showed that when NUCLEONS were bombarded with LEPTONIC probes, the angular distribution for the final state leptons suggested scattering off POINTLIKE constituents inside the nucleon.

(cf RUTHERFORD SCATTERING off NUCLEI.)

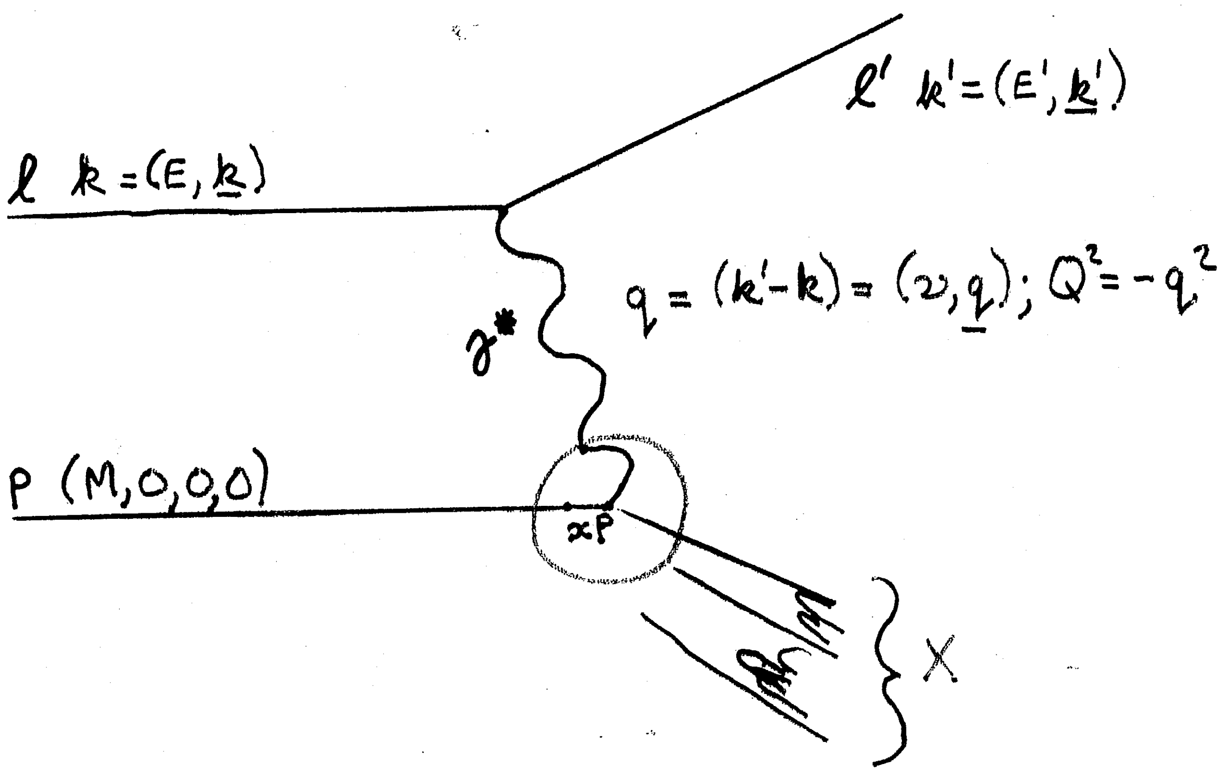
- The interaction kinematics is determined by the LEPTON *initial state* and *final state* 4-vectors alone.

- The nucleon constituents turn out to be FERMIONS (some of them: GLUONS will be discussed later).

The DYNAMICS of the interaction can be framed in terms of the Dirac equation, with some additional development to take into account the momentum distribution of the nucleon constituents (PARTONS).

- STRUCTURE FUNCTIONS describe the momentum distributions for partons of different species. ADDITIONAL structure functions can be defined to specify features such as spin content.

2 KINEMATICS



In general, many different hadronic final states are produced. It is often not possible to detect all the particles in a given hadronic final state, so it is not possible to measure the hadronic 4-vector.

For this reason, conventionally the lepton angles are measured (in fixed target experiments).

If helicities are not taken into account, the parton distributions are described by just TWO independent variables.

It is desirable to make these LORENTZ INVARIANT quantities. Common choices are

$$Q^2 = -q^2$$

$$\nu = \frac{P.q}{M}$$

Note that in the LAB frame more manageable forms for these variables exist.

$$\left.\begin{aligned} \nu &= E' - E \\ Q^2 &= 2EE'(1-\cos\theta) = 4EE'\sin^2\frac{\theta}{2} \end{aligned}\right\} \text{LAB frame}$$

Alternatively, one can use the dimensionless BJORKEN SCALING VARIABLES, x and y. In Lorentz-invariant form, these are

$$x = \frac{Q^2}{2p.q}$$

$$y = \frac{2p.q}{s}$$

Here s is the MANDELSTAM variable

$$s = (p+k)^2 \qquad (\equiv W^2)$$

The corresponding laboratory frame quantities are

$$\left.\begin{aligned} x &= \frac{Q^2}{2M\nu} \\ y &= \nu/E \end{aligned}\right\} \text{LAB Frame}$$

In order to complete this survey of kinematic variables, we note that the Mandelstam variables are defined as

$$s = (p+k)^2 \equiv W^2$$
$$t = (k'-k)^2 \equiv Q^2$$
$$u = (p-k')^2$$

These are related to the previously defined quantities as follows:-

$$s + t + u = M^2 + W^2$$

$$\sin^2 \frac{\theta}{2} = \frac{-tM}{su}$$

$$\nu = \frac{s+u}{2M}$$

$$x = \frac{-t}{s+u}$$

Note that y is proportional to the scattering angle θ^* in the rest frame of the initial lepton and the struck parton

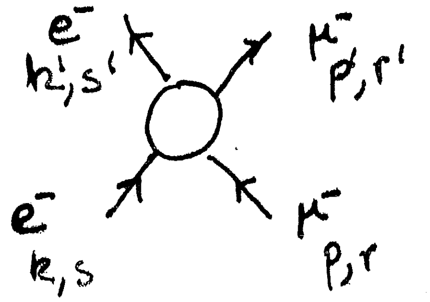
$$\cos \theta^* = (1+y)/2$$

(1) The variable x is normally interpreted as the fraction of the nucleon momentum carried by the struck parton.

(This interpretation is valid only in Lorentz frames where the momentum components are much larger than the nucleon and parton masses.)

SUMMARY OF RESULTS FROM QED

The FEYNMAN rules allow one to calculate amplitudes for fermion-fermion interactions; for $e^-\mu^-$ scattering (so chosen to avoid problems with identical particles and antiparticles), we obtain



$$F_{sr;s'r'} = (-e)\bar{u}(k', s')\gamma_\mu u(k, s)\left(-g^{\mu\nu}/q^2\right)(-e)\bar{u}(p', r')\gamma_\nu u(p, r)$$

The UNPOLARISED cross section is given by

$$d\sigma \sim \frac{1}{4}\sum_{s,r}\sum_{s',r'}\left|F_{sr;s'r'}\right|^2$$

Standard techniques exist for organizing this expression into a product of two traces

$$\frac{1}{4}\sum_{s,r}\sum_{s'r'}|F_{sr;s'r'}|^2 = \left(\frac{e^2}{q^2}\right)^2 \left\{\frac{1}{2}\mathrm{Tr}[(\not{k}'+m)\gamma_\mu(\not{k}+m)\gamma_\nu]\right\}$$
$$\times\left\{\frac{1}{2}\mathrm{Tr}[(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma^\nu]\right\}$$
$$\equiv \left(\frac{e^2}{q^2}\right)^2 L_{\mu\nu}\, M^{\mu\nu}$$

Note that the traces correspond to two *tensors*, one for the electron and one for the muon.

$$L_{\mu\nu} = 2\,[k'_\mu k_\nu + k'_\nu k_\mu + (q^2/2)\, g_{\mu\nu}]$$
$$M^{\mu\nu} = 2\,[p'^\mu p^\nu + p'^\nu p^\mu + (q^2/2)\, g^{\mu\nu}]$$

(performing the traces). Finally the products of 4 vectors are calculated.

(11) In the LAB frame the result of this (laborious) contraction is

$$L_{\mu\nu} M^{\mu\nu} = 16 M^2 E E' \left\{ \cos^2\frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2\frac{\theta}{2} \right\}$$

We now move to the introduction of structure functions in 3 logical steps.

- Scattering from a STRUCTURELESS proton.

  Same form as above. (Still scattering of structureless fermions.)

- Elastic electron proton scattering.

  The matrix element has the same form as before. However the proton tensor is replaced by

$$L^{\mu\nu}_{(p)} = \frac{1}{2} \mathrm{Tr}\, (\not{p}' + m) \Gamma^\mu (\not{p} + m) \Gamma^\nu$$

(ii) The term l_μ contains two arbitrary functions, the *form factors* $F_1(q^2)$, $F_2(q^2)$:

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \kappa \frac{F_2(q^2)}{2M} i\sigma^{\mu\alpha} q_\alpha$$

(cf $\Gamma^\mu \equiv \gamma^\mu$ for a muon)

Contracting the lepton and proton tensors as before, one obtains

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{E'}{E'} \left[\left\{ F_1^2 - \left(\frac{q^2}{4M^2}\right) \kappa^2 F_2^2 \right\} \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right]$$

This is known as the ROSENBLUTH formula for $ep \to ep$ elastic scattering. Note that F_1, F_2 depend on q^2.

(In this formula we have taken into account the FLUX and PHASE SPACE factors,

$$d\sigma = \frac{1}{F} |M|^2 dQ = \frac{1}{|\underline{v}_a - \underline{v}_B|} \cdot \frac{1}{2E_A E_B} \cdot |M|^2 \cdot \frac{d^3 p_c}{(2\pi)^3 2E_c} \frac{d^3 p_d}{(2\pi)^3 2E_p}$$

for $AB \to CD$.)

(15) • Inelastic electron proton scattering.

An alternative and equivalent way of writing the proton tensor, described in AITCHISON & HEY is

$$L^{\mu\nu}_{(p)} = 4A(q^2)\left[p^\mu - ((p.q)/q^2)q^\mu\right]\left[p^\nu - ((p.q)/q^2)q^\nu\right] + 2M^2 B(q^2)\left(-g^{\mu\nu} + q^\mu q^\nu/q^2\right)$$

From this one obtains

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4k^2 \sin^4\frac{\theta}{2}} \cdot \frac{k'}{k} \cdot \left(A\cos^2\frac{\theta}{2} + B\sin^2\frac{\theta}{2}\right)$$

so

$$A(q^2) = F_1^2(q^2) - (q^2/4M^2)\,\kappa^2 F_2^2(q^2)$$

$$B(q^2) = \left(F_1(q^2) + F_2(q^2)\right)^2 \frac{q^2}{2M^2}$$

The second form is more convenient for development in the INELASTIC case. In this case we write a tensor $W^{\mu\nu}(q,p)$

$$W^{\mu\nu}(q,p) = \left(-g^{\mu\nu} + q^\mu q^\nu/q^2\right) W_1(Q^2,\nu) + \left[p^\mu - ((p.q)/q^2)q^\mu\right]\left[p^\nu - ((p.q)/q^2)q^\nu\right] M^{-2} W_2(Q^2,\nu)$$

i.e. the same form as for the electron proton ELASTIC scattering case. Following the same procedures, we obtain

$$\frac{d^2\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\left[W_2 \cos^2\frac{\theta}{2} + 2W_1 \sin^2\frac{\theta}{2}\right]$$

As we are now dealing with INELASTIC scattering Q^2 and ν are now independent variables.

The above formula is GENERAL, and does not build in any model dependence.

Experimentally, it was found that Q^2 and ν were related. Making the substitutions

$$M W_1(Q^2,\nu) \rightarrow F_1(x)$$

$$\nu W_2(Q^2,\nu) \rightarrow F_2(x)$$

with

$$x = Q^2/2M\nu$$

The SAME form for $F_1(x)$, $F_2(x)$ is found for different values of Q^2.

This is known as BJORKEN SCALING. The interpretation of Bjorken scaling is due to Feynman. The key step is to see the fixed ratio of Q^2 to ν as indicative of ELASTIC collisions with nucleon constituents. Comparing directly the fermion-fermion and inelastic electron proton cross sections

$$\frac{d^2\sigma}{dQ^2 d\nu} = \frac{\pi\alpha^2}{4k^2\sin^4(\theta/2)}\frac{1}{kk'}\left(e^2\cos^2\frac{\theta}{2} + e^2\frac{Q^2}{4m^2}\,2\sin^2\frac{\theta}{2}\right)$$

$$\times\,\delta(\nu - Q^2/2m)$$

[FERMION FERMION]

$$\frac{d^2\sigma}{dQ^2 d\nu} = \frac{\pi\alpha^2}{4k^2\sin^4(\theta/2)}\cdot\frac{1}{kk'}\left(W_2\cos^2\frac{\theta}{2} + W_1\,2\sin^2\frac{\theta}{2}\right)$$

[INELASTIC ELECTRON PROTON]

 We obtain the correspondence

$$W_1^i \Rightarrow e_i^2 \frac{Q^2}{4M^2x^2} \delta(\nu - Q^2/2Mx)$$

$$W_2^i \Rightarrow e_i^2 \, \delta(\nu - Q^2/2Mx)$$

for the ith parton species. Here we assume that the proton's mass (and momentum) is shared among the partons, with the struck parton receiving a fraction x:-

$$m = Mx$$

To obtain the full structure function, we

- sum over all parton species
- integrate over all momentum fractions.

Defining $f_i(x)$ as the probability that a parton of species i carries momentum fraction x we obtain

$$W_2(\nu, Q^2) = \sum_i \int dx \, f_i(x) e_i^2 \, \delta(\nu - Q^2/2Mx)$$

We deal with the δ function by using the property

$$\delta(f(x)) = \frac{\delta(x-x_0)}{\left| df/dx \right|_{x=x_0}}$$

where

$$f(x_0) = 0$$

Thus

$$\delta(\nu - Q^2/2Mx) = \frac{x}{\nu}\, \delta(x - Q^2/2M\nu)$$

Substituting,

$$W_2(\nu, Q^2) = \sum_i \int dx\, f_i(x)\, e_i^2 \frac{x}{\nu}\, \delta(x - Q^2/2M\nu)$$

$$= \frac{1}{\nu} \sum_i e_i^2\, x\, f_i(x) \qquad \left(x \text{ given by } \frac{Q^2}{2M\nu}\right)$$

$$\boxed{\nu W_2(\nu, Q^2) = \sum_i e_i^2\, x\, f_i(x) \equiv F_2(x)}$$

2 points:-

(i) "Deep inelastic" behaviour emerges as Q^2 increases, leading to a region where Bjorken scaling holds. Q^2 can be thought of as setting the de Broglie wavelength of the virtual photon. At low Q^2 the proton is seen as structureless. As Q^2 increases the structure of the proton is revealed.

(ii) The formalism has been developed using correspondence between functional dependences in pointlike and substructure cases. The *form* of the structure functions remains unknown and must be determined by experiment.

THE CALLAN-GROSS RELATION

The same steps can be followed for the structure function W_1 :-

$$W_1^i = e_i^2 \frac{Q^2}{4M^2x^2} \delta(\nu - Q^2/2Mx)$$

so

$$W_1(\nu, Q^2) = \sum_i \int dx\, f_i(x) e_i^2 \frac{Q^2}{4M^2x} \delta(\nu - Q^2/2Mx)$$

Using

$$\delta(\nu - Q^2/2Mx) = \frac{x}{\nu} \delta(x - Q^2/2M\nu)$$

we obtain

$$W_1(\nu, Q^2) = \sum_i \int dx\, f_i(x) e_i^2 \frac{Q^2}{4M^2x} \cdot \frac{1}{\nu} \delta(x - Q^2/2M\nu)$$

$$= \sum_i \frac{Q^2}{4M^2\nu} \cdot \frac{2M\nu}{Q^2} \cdot e_i^2 \cdot f_i(x) = \frac{1}{2M} \sum_i e_i^2 f_i(x)$$

As $\quad W_1(\nu, Q^2) \equiv \dfrac{F_1(x)}{M}$

$$\boxed{F_1(x) = \tfrac{1}{2} \sum_i e_i^2 f_i(x)}$$

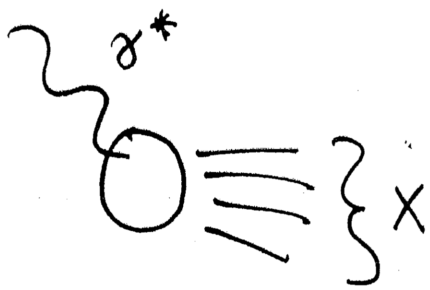
Comparing this with the previously obtained expression for $F_2(x)$, we obtain

$$\boxed{F_2(x) - 2x F_1(x) = 0}$$

This is known as the CALLAN-GROSS relation. As it was derived through the matching of functional dependence with fermion-fermion scattering, it is a consequence of the spin-½ nature of the constituents.

EXPERIMENTAL TESTS

The usual test for the Callan Gross relation is to study the cross section ratio $R = \frac{\sigma_S}{\sigma_T}$, where σ_S (σ_T) is the cross section for deep inelastic scattering via longitudinally (transversely) polarized virtual photons.



$\gamma p \to X$

Our starting point is the cross section for polarization λ:-

$$\sigma_\lambda(\gamma^* p \to X) = \frac{4\pi^2\alpha}{K} \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda) W^{\mu\nu}$$

[AITCHISON & HEY p. 96]

The $\epsilon_\mu(\lambda)$ are *polarization vectors*.

For virtual photons the flux factor K is ARBITRARY. The normal procedure is to put

$$K = (W^2 - M^2)/2M$$

as for real photons. K corresponds to the energy of the REAL photon needed to create the state X.

This is known as the HAND convention.

The polarization vectors are

$$\epsilon^\mu(\lambda=\pm 1) = \mp 2^{-1/2}(0,1,\pm i,0)$$

$$\epsilon^\mu(\lambda=0) = \frac{1}{\sqrt{Q^2}}(q^3, 0, 0, q^0)$$

The *transverse* polarization cross section is the sum of the $\lambda=+1$ and $\lambda=-1$ contributions

$$\sigma_T = \left(\frac{4\pi^2\alpha}{K}\right)\frac{1}{2}\sum_{\lambda=\pm 1}\epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda)\, W^{\mu\nu}$$

The proton tensor is

$$W^{\mu\nu} = (-g^{\mu\nu} + q^\mu q^\nu/q^2)W_1(Q^2,\nu) + [p^\mu - ((p.q)/q^2)q^\mu][p^\nu - ((p.q)/q^2)q^\nu]M^{-2}W_2(Q^2,\nu)$$

and, in the LAB frame

$$p^\mu = [M,0,0,0]$$
$$q^\mu = [q^0,0,0,q^3]$$

so

$$\sigma_T = \left(\frac{4\pi^2\alpha}{K}\right)W_1$$

$$\sigma_L = \left(\frac{4\pi^2\alpha}{K}\right)\epsilon^*_\mu(\lambda=0)\epsilon_\nu(\lambda=0)W^{\mu\nu} = \left(\frac{4\pi^2\alpha}{K}\right)\left[\left(1+\frac{\nu^2}{Q^2}\right)W_2 - W_1\right.$$

(23) Converting to F_1 and F_2

$$\nu W_2 \rightarrow F_2$$
$$M W_1 \rightarrow F_1$$

so in the "deep inelastic limit", Q^2 and ν large with $x = Q^2/2M\nu$ finite

$$\sigma_T \rightarrow \frac{4\pi^2\alpha}{MK} F_1$$
$$\sigma_S \rightarrow \frac{4\pi^2\alpha}{MK}\left(\frac{1}{2x}\right)(F_2 - 2xF_1)$$

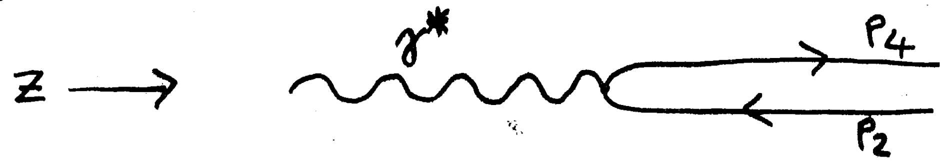
Thus as Q^2 becomes large, the Callan-Gross relation yields

$$R = \frac{\sigma_S}{\sigma_T} \rightarrow 0$$

A current algebra calculation for pointlike spin 0 partons yields

$$\frac{\sigma_T}{\sigma_S} \rightarrow 0$$

Thus to test whether the partons are spin 0 or spin ½ one tests whether R tends to 0 or ∞ as Q^2 increases.



$$q + p_2 = p_4$$

$$q = (0, 0, 0, (Q^2)^{1/2})$$
$$p_2 = (E_B, 0, 0, -(Q^2)^{1/2}/2)$$
$$p_4 = (E_B, 0, 0, +(Q^2)^{1/2}/2)$$

Consider photon-parton scattering in the BREIT frame. In this frame (also known as the BRICK WALL frame)

- the virtual photon has ZERO ENERGY
- the parton reverses direction in the collision.

Consider ANGULAR MOMENTUM conservation.

- A SPIN ZERO parton cannot absorb angular momentum, but a transversely polarized incoming photon brings one unit of angular momentum, so $\sigma_T \to 0$ as $Q^2 \to \infty$.
- A SPIN ½ parton absorbs one unit of angular momentum by reversing its spin, so $\sigma_T \neq 0$ as $Q^2 \to \infty$.

Experimentally, the ratio R is small but non-zero. This is taken to show that partons are spin ½ objects

Recently, interest has focussed on deviations from the Callan-Gross prediction. Three possible effects are listed below.

(i) "Target Mass" Effects. If partons are not massless but have an intrinsic mass m, R receives a contribution

$$R_{TM} \sim 4m^2/Q^2$$

(ii) Intrinsic k_T. k_T is the intrinsic transverse momentum for partons inside the proton, and would give a contribution

$$R_{k_T} = 4\langle k_T\rangle/Q^2$$

(iii) Higher Twist effects. These arise from binding effects between quarks in a proton, e.g. DIQUARKS. They give rise to contributions of the form

$$R_{HT} = \mu(\nu, Q^2)/Q^2$$

Thus all the corrections point to a $1/Q^2$ increase in R as Q^2 becomes small.

1

DEVELOPMENT OF THE QUARK PARTON MODEL

SUMMARY

The main results obtained so far are:-

(i) The proton (nucleon) consists of *pointlike* scattering centres, called PARTONS.

[BJORKEN SCALING]

(ii) The partons (or some of them) are SPIN-1/2 OBJECTS.

[CALLAN-GROSS RELATION, σ_T/σ_L as $Q^2 \to \infty$]

Considerably more can be learned fron a suitable choice of beam and target.

(2) The MASTER FORMULA of the Quark Parton model is

$$\boxed{F_2^{\ell N}(x) = \sum_i e_i^2 \, x f(x)}$$

Expanding this out, we obtain

$$F_2^{\ell p}(x) = x\left\{\frac{4}{9}\left[u(x)+\bar{u}(x)+c(x)+\bar{c}(x)\right] + \frac{1}{9}\left[d(x)+\bar{d}(x)+s(x)+\bar{s}(x)\right]\right\}$$

for (charged) lepton-PROTON interactio[n]

What about NEUTRON targets?

In principle, one could rewrite the expression for F_2, noting that now the expressions $u_n(x)$, $d_n(x)$ are different from those ($u_p(x)$ and $d_p($ for scattering off protons.

(3) However, the x-distributions for the u and d quarks in neutrons and protons are related by ISOSPIN INVARIANCE.

Thus

$$u_p(x) = d_n(x) \equiv u(x)$$

$$d_p(x) = u_n(x) \equiv d(x)$$

i.e. we use the proton names to labe the nucleon quark distributions.

With this convention, one has, for ℓn scattering,

$$F_2^{\ell n}(x) = x\Big\{\tfrac{4}{9}\left[d(x)+\bar{d}(x)+c(x)+\bar{c}(x)\right]$$
$$+\tfrac{1}{9}\left[u(x)+\bar{u}(x)+s(x)+\bar{s}(x)\right]\Big\}$$

so that the *ratio* of structure functions is given by.

④

$$\frac{F_2^{\ell n}(x)}{F_2^{\ell p}(x)} = \frac{\frac{4}{9}[d(x)+\bar{d}(x)+c(x)+\bar{c}(x)] + \frac{1}{9}[u(x)+\bar{u}(x)+s(x)+\bar{s}(x)]}{\frac{4}{9}[u(x)+\bar{u}(x)+c(x)+\bar{c}(x)] + \frac{1}{9}[d(x)+\bar{d}(x)+s(x)+\bar{s}(x)]}$$

With no further detailed information, we see the ratio is bounded:-

(a) If the u quark distribution dominates,
$F_2^{\ell n}(x)/F_2^{\ell p}(x) \to 1/4$

(b) If the d quark distribution dominates,
$F_2^{\ell n}(x)/F_2^{\ell p}(x) \to 4$

(c) If NEITHER $u(x)$ nor $d(x)$ dominates the ratio takes *intermediate* values. If either $s(x)$ or $c(x)$ dominate,

$F_2^{\ell n}(x)/F_2^{\ell p}(x) \to 1.$

Thus

$$\frac{1}{4} \leqslant \frac{F_2^{\ell n}(x)}{F_2^{\ell p}(x)} \leqslant 4$$

It is instructive to separate the u and d quark distributions.

Not all scattering takes place off VALENCE quarks, i.e. those whose abundance is governed by the static SU(3) quark model. In addition there is a "SEA" of *off-mass-shell* $q\bar{q}$ pairs (of all flavours).

Thus

$$u(x) = u_v(x) + u_s(x)$$

$$d(x) = d_v(x) + d_s(x)$$

While all that is *strictly* required for the sea quarks is that $q_{s:j}(x) = \bar{q}_{s:j}(x)$

⑥

However, it is usually assumed that the SEA quark distributions for u, d and s quarks are ALL equal:

$$u_s(x) = \bar{u}_s(x) = d_s(x) = \bar{d}_s(x) = s(x) = \bar{s}(x) \; [= c(x) = \bar{c}(x)$$

More accurately, the s quark distributions *should* be different from those for u and d quarks owing to the mass of the s quark.

The c quark contributions are a more extreme case; they are negligible unless Q^2 is well above charm production thresho

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Using these relations, it is possible to restrict the bounds on the ratio $F_2^{\ell n}/F_2^{\ell p}$ further.

(7) If the SEA contribution dominates

$$\frac{F_2^{\ell n}}{F_2^{\ell p}} \rightarrow 1$$

[The SEA is largest at small x, $x \to 0$.]

If the VALENCE contribution dominates

$$\frac{F_2^{\ell n}}{F_2^{\ell p}} \rightarrow \frac{2}{3} \text{ to } \frac{1}{4}$$

[The VALENCE contribution is stronges for $x \sim 0.3$.]

(8)

SUM RULES

Constraints on the relative magnitudes of quark distribution functions are often expressed in terms of SUM RULES. These are *integral equations* of the form

$$\int_0^1 g(x)\,dx = K$$

$g(x)$ is a sum/difference of quark distribution functions or structure functions.

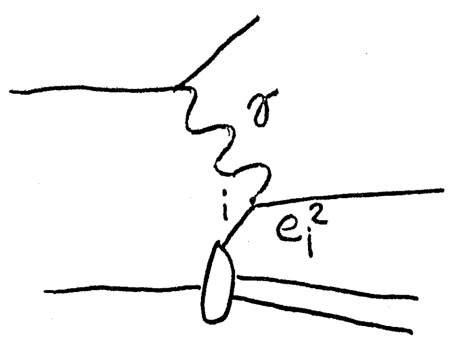
For example, the requirement that net strangeness be zero yields

$$\int_0^1 (s(x) - \bar{s}(x))\,dx = 0$$

9

The quark distribution functio[n]
$q_i(x)$ are not directly accessible to experiment. Structure functions give linear combinations of the $q_i(x)$.

Scattering with charged lepto[ns] uses a photon as an exchange particle. The coupling is proportio[nal] to the square of the quark charges.



Scattering with different probes gives rise to different couplings, allowin[g] separation of quark distribution functio[ns]