SYMMETRIES IN PARTICLE PHYSICS

LECTURE 1

The application of SYMMETRY PRINCIPLES has been a powerful tool in the development of particle physics.

When formulating physical laws, the requirement that they display symmetry (sometimes to very high order) has been fruitful.

HOW?

There is a connection between symmetries and conserved quantities

- Especially useful in particle physics, where only FINAL STATES (after all interactions have ceased) are measurable.

2. EXAMPLE: Spatial translation and momentum conservation.

[PERKINS Ch. 3, WEYL]

In a quantum mechanical formalism, invariance under spatial translation means that the eigenvalues of a system are unchanged by the translation. For the Hamiltonian

$$HD = DH$$

where D represents the spatial translation (displacement)

Note that, in the HEISENBERG representation, an operator which commutes with the Hamiltonian has eigenvalues which are invariant in time

$$\frac{d}{dt} D = \frac{i}{\hbar} [H, D]$$

3.

The operator D for finite displacements can be built up from repeated applications of the infinitesimal translation

$$D = 1 + \delta r \frac{\partial}{\partial r}$$

so

$$D \psi(r) = \psi(r + \delta r)$$

as required

$\partial/\partial r$ is the generator of the infinitesimal translation. It is related (up to a constant) to the momentum operator

$$p = -i\hbar \frac{\partial}{\partial r}$$

In other words, invariance under spatial translation implies

$$[H, p] = 0$$

i.e. momentum conservation.

What was used in reaching this conclusion?

- Infinitesimal transformations, their operator representations, and the generators of the transformation.

- Commutation relations among operators, in this case with the Hamiltonian. (The Hamiltonian plays a special rôle, because of its connection with time development.)

ANGULAR MOMENTUM

Angular momentum algebra is the most familiar variant of a formalism describing

- Orbital angular momentum
- Spin
- Isotopic Spin

$\Big\}$ su(2)

and (with some generalization

- colour
- flavour

$\Big\}$ su(3)

The angular momentum Hilbert space is spanned by eigenstates of two operators, J^2 and J_z :-

$$J^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle$$

$$J_z |j,m\rangle = m\hbar |j,m\rangle$$

There are also raising and lowering operators,

$$J_\pm |j,m\rangle = J_x + iJ_y |j,m\rangle = C_{jm} |j, m\pm 1\rangle$$

ALL these relations can be derived from the basic commutation relation between angular momentum components.

$$\boxed{[J_j, J_k] = i\epsilon_{jkl} J_l}$$

$\epsilon_{123} = +1$ {and cyclic permutations}

$\epsilon_{321} = -1$ {and cyclic permutations}

$\epsilon_{jkl} = 0$ otherwise

---

1. **State vectors can be simultaneous eigenstates of J^2 and J_3.**

$$J^2 = J_1^2 + J_2^2 + J_3^2$$

$$\begin{aligned}
J^2 J_3 &= J_1^2 J_3 + J_2^2 J_3 + J_3^3 \\
&= J_1\{J_3 J_1 - iJ_2\} + J_2\{J_3 J_2 + iJ_1\} + J_3^3 \\
&= J_1 J_3 J_1 - iJ_1 J_2 + J_2 J_3 J_2 + iJ_2 J_1 + J_3^3 \\
&= \{J_3 J_1 - iJ_2\} J_1 - iJ_1 J_2 + \{J_3 J_2 + iJ_1\} J_2 + iJ_2 J_1 + J_3^3 \\
&= J_3 J_1^2 - iJ_2 J_1 - iJ_1 J_2 + J_3 J_2^2 + iJ_1 J_2 + iJ_2 J_1 + J_3^3 \\
&= J_3 [J_1^2 + J_2^2 + J_3^2]
\end{aligned}$$

$$[J^2, J_3] = 0$$

Assumption: operators J_i are Hermitian. {Observables}

2. The eigenvalues of the square of a Hermitian operator are positive semi-definite (non-negative).

Notation. Write inner product as $(\psi, Q^2\psi)$

Define $\int \psi^* A\phi \, d\tau = (\psi, A\phi)$

While in general $(\psi, A^\dagger\phi) = (A\psi, \phi)$

only for HERMITIAN operators, where $A = A^\dagger$ do we have

$$(A\psi, \phi) = (\psi, A\phi)$$

DIRAC notation, $\langle\psi|A|\phi\rangle$, does not specify whether operator acts to L or R, which is safe only for HERMITIAN operators.

$$(\psi, Q^2\psi) = (\psi, Q^\dagger Q\psi) = \sum_n (\psi, Q^\dagger \phi_n)(\phi_n, Q\psi)$$

$$= \sum_n |(\phi_n, Q\psi)|^2 \geqslant 0$$

The ϕ_n are an orthonormal set of states spanning the Hilbert space.

Thus

$$J^2\psi = \kappa^2\psi \qquad (\kappa \text{ real})$$

3. $\boxed{\text{The eigenvalues of } J_3 \text{ are BOUNDED.}}$

Assumption: J_3 satisfies an eigenvalue equation.

$$J_3 \psi_m = m \psi_m$$

Consider $\quad J_1^2 + J_2^2 = J^2 - J_3^2$. LHS is UNKNOWN, but is made up from squares of Hermitian operators, so its eigenvalues must be non-negative.

Then

$$(J^2 - J_3^2)\psi_m = (K^2 - m^2)\psi_m$$

so

$$K^2 - m^2 \geqslant 0$$

$$|m| \leqslant K.$$

4. $\boxed{\text{Demonstrate properties of ladder operators.}}$

Ladder operators are

$$J_+ = J_1 + i J_2$$

$$J_- = J_1 - i J_2$$

To find J_3 eigenvalue, consider

$$J_3 (J_1 + i J_2)\psi_m = (J_3 J_1 + i J_3 J_2)\psi_m$$

$$= [(J_1 J_3 + i J_2) + i(J_2 J_3 - i J_1)]\psi_m$$

$$= [(J_1 + i J_2) J_3 + (J_1 + i J_2)]\psi_m = (J_1 + i J_2)(J_3 + 1)\psi_m = (J_1 + i J_2)(m+1)\psi_m$$

So

$$J_3\{(J_1+iJ_2)\psi_m\} = (m+1)\{(J_1+iJ_2)\psi_m\}$$

i.e.

$$(J_1+iJ_2)\psi_m = C\psi_{m+1}$$

- Similarly $\quad J_-\psi_m = C'\psi_{m-1}$

$$\boxed{\text{... and determine commutation relations}}$$

$$\begin{aligned} J_+J_- &= J_1^2 + iJ_2J_1 - iJ_1J_2 + J_2^2 \\ &= J_1^2 + J_2^2 - i(J_1J_2 - J_2J_1) \\ &= J_1^2 + J_2^2 + J_3 \end{aligned}$$

similarly

$$\begin{aligned} J_-J_+ &= J_1^2 - iJ_2J_1 + iJ_1J_2 + J_2^2 \\ &= J_1^2 + J_2^2 - J_3 \end{aligned}$$

so

$$\boxed{[J_+, J_-] = 2J_3}$$

5. $\boxed{\text{Correlate coefficients } \kappa^2 \text{ and } m}$

m is bounded [3], so there is a highest state m_0 such that

$$J_+\psi_{m_0} = 0$$

10. So

$$J_-\{J_+\psi_{m_0}\} = 0$$

$$\left[J_- J_+ = J_1^2 + J_2^2 - J_3 = J^2 - J_3^2 - J_3\right]$$

$$(J^2 - J_3^2 - J_3)\psi_{m_0} = 0$$

$$(K^2 - m_0^2 - m_0)\psi_{m_0} = 0$$

so

$$K^2 = m_0(m_0+1)$$

Similarly, using J_- and m_1, the LOWEST eigenvalue

$$K^2 = m_1(m_1 - 1)$$

so

$$m_0(m_0+1) = m_1(m_1-1)$$

Solutions are

$$m_0 = -m_1 \qquad (a)$$

$$m_1 = m_0 + 1 \qquad (b)$$

(b) contradicts hypothesis that m_0 is largest eigenvalue

so

$$m_0 = -m_1$$

So eigenstates run from m_0 to $-m_0$ in integer steps. Define $j = m_0$, then

$$\boxed{J^2\psi_{jm} = j(j+1)\psi_{jm}}$$

Determine coefficients $C^{\pm}_{jm}$ for Ladder Operators

Define

$$J_+ \psi_{jm} = C^+_{jm} \psi_{j\,m+1}$$
$$J_- \psi_{jm} = C^-_{jm} \psi_{j\,m-1}$$

C^+_{jm} and C^-_{jm} are related:-

$$(\psi_{j\,m+1}, J_+ \psi_{jm}) = C^+_{jm}$$

Now $J_- = J_+^\dagger$, so

$$(J_- \psi_{j\,m+1}, \psi_{jm}) = C^+_{jm}$$

and

$$(\psi_{jm}, J_- \psi_{j\,m+1}) = C^-_{j\,m+1}$$

so

$$\boxed{C^+_{jm} = C^{-*}_{j\,m+1}}$$

Now consider

$$\begin{aligned} J_+ J_- \psi_{jm} &= J_+ C^-_{jm} \psi_{j\,m-1} \\ &= C^+_{j\,m-1} C^-_{jm} \psi_{jm} \\ &= C^{-*}_{jm} C^-_{jm} \psi_{jm} \\ &= |C^-_{jm}|^2 \psi_{jm} \end{aligned}$$

12. This is an EIGENVALUE equation. We can rewrite $J_+ J_-$ as

$$J_+ J_- = J_1^2 + J_2^2 + J_3 = J^2 - J_3^2 + J_3$$

so

$$J_+ J_- \psi_{jm} = [j(j+1) - m^2 + m]\psi_{jm} = |C^-_{jm}|^2 \psi_{jm}$$

i.e.

$$\boxed{|C^-_{jm}|^2 = j(j+1) - m(m-1)}$$

Now $|C^+_{j\,m-1}|^2 = |C^-_{jm}|^2$, so

$$|C^+_{j\,m-1}|^2 = j(j+1) - m(m-1)$$

$$\boxed{|C^+_{jm}|^2 = j(j+1) - m(m+1)}$$

This fixes $|C^{\pm}_{jm}|^2$. PHASES are chosen by convention. We normally choose the POSITIVE SQUARE ROOT. This is the CONDON & SHORTLEY phase convention.

INTRODUCTION TO LIE ALGEBRAS

Recap:

Angular momentum relations were "recovered" from

$$[J_j, J_k] = i\epsilon_{jkl} J_l$$

which is a specific case of a LIE ALGEBRA

A LIE ALGEBRA is formed from a set of operators X_ρ such that

$$[X_\rho, X_\sigma] = \sum_\tau C^\tau_{\rho\sigma} X_\tau$$

where the $C^\tau_{\rho\sigma}$ are constants (structure constants).

In each Lie algebra, there will be one or more operators C_μ (CASIMIR operators) which commute with ALL the basic operators C_μ.

$$[C_\mu, X_\rho] = 0 \quad \text{for all } \mu, \rho.$$

2.

For each LIE ALGEBRA there is a corresponding LIE GROUP, a continuous group of transformations. The operators X_δ are the generators of the Lie group, i.e. the infinitesimal transformation

$$\psi' = (1 + i\epsilon X_\delta)\psi$$

is a transformation in the Lie group.

TRANSFORMATIONS

Transformations among VARIABLES involve replacing each of a list of variables by a new value based on the previous variable list.

$$x' = f(x, y, z) \qquad \text{e.g.} \quad x' = x + a_x$$
$$y' = g(x, y, z) \qquad \qquad y' = y + a_y$$
$$z' = h(x, y, z) \qquad \qquad z' = z + a_z$$

N.B. there are "discrete" transformations

$$x' = -x$$
$$y' = -y$$
$$z' = -z$$

3. For each transformation, we need to define a corresponding operator U for the quantum mechanical state vector ψ

$$\psi' = U\psi$$

The transformed state vector should have the SAME NORMALIZATION as the initial state vector.

This determines that the transformation operators should be UNITARY.

Let

$$(\psi, \psi) = 1.$$

We want

$$(\psi', \psi) = 1$$

But

$$(\psi', \psi') = (U\psi, U\psi) = (\psi, U^{\dagger}U\psi)$$

so

$$U^{\dagger}U = 1$$

Right-multiplying by U^{-1}

$$U^{\dagger} = U^{-1}$$

which is the definition of a unitary operator.

4\. Now consider the EXPECTATION VALUE of a Hermitian operator. The numerical value of the expectation value should be unchanged.

$$(\psi', A'\psi) = (\psi, A\psi)$$

$$(U\psi, A'U\psi) = (\psi, A\psi)$$

$$(\psi, U^{\dagger}A'U\psi = (\psi, A\psi)$$

i.e.

$$U^{\dagger}A'U = A$$

To invert we - left multiply by U
- right multiply by $U^{\dagger}$

$$UU^{\dagger}A'UU^{\dagger} = UAU^{\dagger}$$

$$A' = UAU^{\dagger}$$

Note that if A is invariant under the transformation

$$A' = A$$

$$UAU^{\dagger} = A$$

$$UA = AU \quad \text{or} \quad [U,A] = 0$$

If $\mathcal{U}$ does not involve TIME, it will commute with the Hamiltonian

$$[\mathcal{U}, H] = 0$$

Note however that, as $\mathcal{U}$ is a UNITARY operator it does not correspond to an OBSERVABLE: observables are represented by HERMITIAN operators.

The GENERATORS of continuous transformations represented by unitary operators are HERMITIAN.

Let the generator of an infinitesimal unitary transformation $\mathcal{U}$ be the operator G

$$\mathcal{U} = 1 + i\,\delta\alpha\, G$$

Then

$$\mathcal{U}^\dagger = 1 - i\,\delta\alpha\, G^\dagger$$

so

$$\mathcal{U}^\dagger\mathcal{U} = (1 - i\,\delta\alpha\, G^\dagger)(1 + i\,\delta\alpha\, G)$$

$$= (1 + i\delta\alpha\,(G - G^\dagger) + O(\delta\alpha)^2$$

i.e.

$$G = G^\dagger$$

ROTATIONS

An infinitesimal rotation (e.g. about the 3-axis) is generated by

$$U_\epsilon = 1 + i\epsilon J_3$$

A finite transformation is built up by a succession of infinitesimal transformations

$$U(\theta) = [U(\epsilon)]^n = \left(1 - i\frac{\theta}{n} J_3\right)^n \xrightarrow[n\to\infty]{} e^{-i\theta J_3}$$

A *general* rotation is made up from three rotations about independent axes.

(i) a rotation α about the z axis, $e^{-i\alpha J_3}$

(ii) a rotation β about the y' axis, $e^{-i\beta J_2'}$

(iii) a rotation γ about the z'' axis, $e^{-i\gamma J_3''}$

Note that while, as defined, each rotation is performed with respect to a set of axes Σ', Σ'' resulting from the previous transformations, in practice the rotations can be thought of as being performed about the ORIGINAL axes.

8.

Remember that

$$A' = U A U^{\dagger}$$

so

$$e^{-i\beta J_2'} = e^{-i\alpha J_3} e^{-i\beta J_2} e^{i\alpha J_3}$$

$$e^{-i\gamma J_3''} = e^{-i\beta J_2'} e^{-i\alpha J_3} e^{-i\gamma J_3} e^{i\alpha J_3} e^{i\beta J_2'}$$

so

$$e^{-i\gamma J_3''} e^{-i\beta J_2'} e^{-i\alpha J_3}$$

$$= \left[e^{-i\beta J_2'} e^{-i\alpha J_3} e^{-i\gamma J_3} \cancel{e^{+i\alpha J_3}} \cancel{e^{i\beta J_2'}} \right] \cancel{e^{-i\beta J_2'}} \cancel{e^{-i\alpha J_3}}$$

$$= \left[e^{-i\alpha J_3} e^{-i\beta J_2} \cancel{e^{i\alpha J_3}} \right] \cancel{e^{-i\alpha J_3}} e^{-i\gamma J_3}$$

$$= e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}$$

Note that the ORDER of the rotations has been INVERTED.

The standard eigenstates ψ_{jm} are eigenstates of J_3, so they are eigenstates of $e^{-i\gamma J_3}$

$$e^{-i\gamma J_3} \psi_{jm} = \left(1 - i\gamma J_3 - \frac{\gamma^2}{2!} J_3^2 \ldots\right) \psi_{jm}$$

$$= e^{-i\gamma m} \psi_{jm}$$

Thus the FIRST and THIRD rotations are simple exponential factors.

The second rotation is more complicated as the ψ_{jm} are not eigenstates of J_2.

In general we may write

$$e^{-i\beta J_2}\psi_{jm} = \sum_{m'=-j}^{m'=j} \psi_{jm'}\, d^{(j)}_{m'm}(\beta)$$

We may thus represent the rotation operator as

$$U(R(\alpha,\beta,\gamma))\psi_{jm} = \sum_{m'=-j}^{m'=j} \psi_{jm'}\, \mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma)$$

where

$$\mathcal{D}^{(j)}_{m'm'}(\alpha,\beta,\gamma) = e^{-im'\alpha}\, d^{(j)}_{m'm}(\beta)\, e^{-im\gamma}$$

(α,β,γ) are the EULER ANGLES for the rotation.

General formulae exist for the $d^{(j)}_{m'm}(\beta)$:-

$$d^{(j)}_{mn}(\beta) = \sum_t (-1)^t \frac{[(j+m)!(j-m)!(j+n)!(j-n)!]^{1/2}}{(j+m-t)!(j-n-t)!\,t!\,(t+n-m)!}$$

$$\times \left(\cos\beta/2\right)^{2j+m-n-2t}\left(\sin\frac{\beta}{2}\right)^{2t+n-m}$$

For numerical work, use (e.g.) FUNCTION DSMALL(AJ,AM,AN,BETA) in CERN library GENLIB.

MATRIX REPRESENTATIONS

It is convenient to find a mathematical construct which satisfies the commutation relations.

MATRICES, which in general do not commute, can be chosen to make such a representation.

For SPIN, the simplest such system is made up by the PAULI MATRICES:-

$$\underline{\sigma_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \underline{\sigma_2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \underline{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where

$$J_i = \tfrac{1}{2}\hbar \underline{\sigma_i}$$

and the matrices act on two dimensional column vectors

$$|J_3 = +1/2\rangle \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; |J_3 = -1/2\rangle \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

known as SPINORS.

11. The Pauli matrices are

(i) traceless (i.e. $\mathrm{Tr}\,\sigma = 0$)

(ii) Hermitian

The unitary matrices which they generate have determinant +1.

We have

$$U = e^{i\epsilon \underline{\tilde{G}}}$$

and

$$\det|e^{\underline{A}}| = e^{\mathrm{tr}\,\underline{A}}$$

Note also that as

$$\det \underline{A} \det \underline{B} = \det \underline{AB}$$

the product of two matrices with determinant 1 will ALSO have determinant 1 i.e. such matrices form a group under multiplication. (Inverses and unit matrices can be found.)

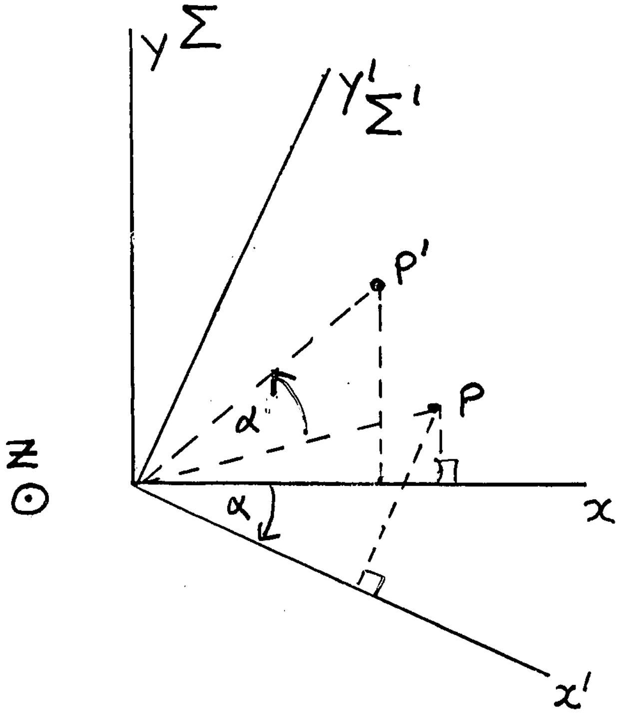
12.

The group of 2×2 matrices having determinant ± 1 is known as the $U(2)$ group. If the determinant is $+1$ then we have the $SU(2)$ group.

Any group in one-to-one correspondence with the $SU(2)$ group (e.g. the group generated by the angular momentum operators) is ALSO an $SU(2)$ group.

FINITE ROTATIONS

I DEFINITIONS: "ACTIVE" AND "PASSIVE" ROTATIONS.



STANDARD (PASSIVE) ROTATION

$$x' = x\cos\alpha - y\sin\alpha$$
$$y' = x\sin\alpha + y\cos\alpha$$

In a PASSIVE rotation Z_α, the coordinate system moves CLOCKWISE by α to new system Σ'; we recompute coordinates of a point P in Σ'.

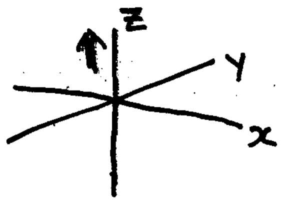
In an ACTIVE rotation Z'_α, the coordinate system does not move; instead, the point P is rotated ANTICLOCKWISE by α to P′.

The coordinates of P in Σ' are the same as those of P′ in Σ.

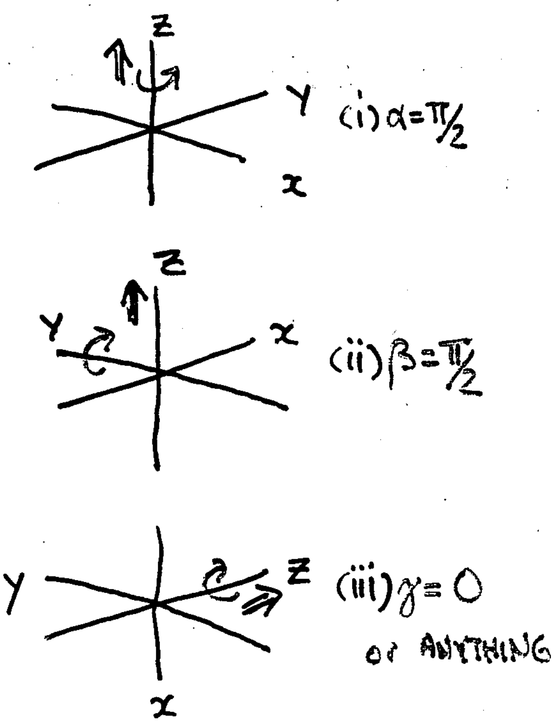
For 3-D rotations, we take the ACTIVE interpretation of a rotation.

EXAMPLE 1. ROTATE SPIN-½ SYSTEM DIRECTED ALONG +Z SO IT POINTS ALONG +Y.

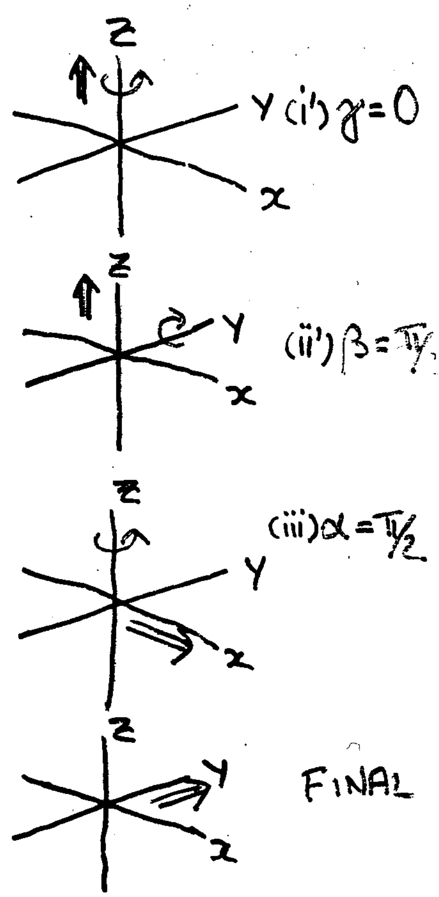
ORIGINAL



ROTATIONS w.r.t. MOVING AXES



ROTATIONS w.r.t. ORIGINAL AXES



FINAL

Rotation is

$R(\pi/2, \pi/2, 0)$

SPIN-½ ROTATION MATRIX IS

$$\begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}$$

so we obtain

$$\begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

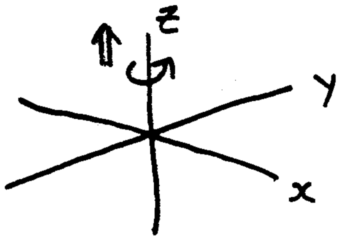
$$= \frac{1}{2}\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \underline{\underline{\frac{1}{2}\left\{\begin{pmatrix} 1-i \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1+i \end{pmatrix}\right\}}} \equiv \phi'$$

CHECK: $\sigma_y \phi' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2}\left\{\begin{pmatrix} 1-i \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1+i \end{pmatrix}\right\} = \frac{1}{2}\left\{\begin{pmatrix} 0 \\ 1+i \end{pmatrix} + \begin{pmatrix} 1-i \\ 0 \end{pmatrix}\right\} = \phi'$ OK

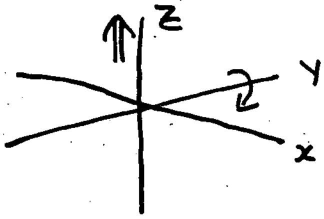
EXAMPLE 2: ROTATE SPIN-$\frac{1}{2}$ SYSTEM DIRECTED ALONG $+z$ SO IT POINTS ALONG $+x$

Rotation is

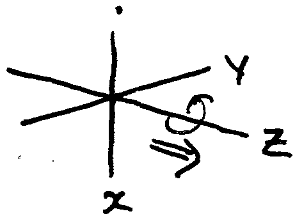
$R(0, \pi/2, 0)$



(i) $\alpha = 0$



(ii) $\beta = \pi/2$



(iii) $\gamma = 0$

ROTATION

$$\underline{\underline{1}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \underline{\underline{1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \equiv \phi''$$

CHECK: $\sigma_x \phi'' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \phi''$

1.

SU(3)

SU(3) is the group formed under multiplication by 3x3 matrices with determinant +1.

Its generators should be 3x3 traceless matrices.

The general 3x3 traceless matrix can be written

$$\underline{M} = \frac{1}{2}\begin{pmatrix} a_3 + 3^{-1/2}a_8 & a_1 - ia_2 & a_4 - ia_5 \\ a_1 + ia_2 & -a_3 + 3^{-1/2}a_8 & a_6 - ia_7 \\ a_4 + ia_5 & a_6 + ia_7 & -\frac{2}{\sqrt{3}}\cdot a_8 \end{pmatrix}$$

This may be written in the form

$$\underline{M} = \tfrac{1}{2}(a_1\underline{\lambda}_1 + a_2\underline{\lambda}_2 + \dots + a_8\underline{\lambda}_8)$$

The 8 λ_i are the GELL MANN matrices.

2. Then, by inspection, the Gell-Mann matrices are

$$\underline{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \underline{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \underline{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underline{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \underline{\lambda}_8 = 3^{-1/2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

By inspection, it is possible to show that the $\underline{\lambda}_i$ satisfy

$$[\underline{\lambda}_i , \underline{\lambda}_j] = 2i \sum_{k=1}^{k=8} f_{ijk} \underline{\lambda}_k$$

i.e. the commutation relation defining a LIE ALGEBRA.

The STRUCTURE CONSTANTS, f_{ijk}, are found to be

3.

i	j	k	f_{ijk}	i	j	k	f_{ijk}
1	2	3	1	3	4	5	$1/2$
1	4	7	$1/2$	3	6	7	$-1/2$
1	5	6	$-1/2$	4	5	8	$1/2 \cdot 3^{1/2}$
2	4	6	$1/2$	6	7	8	$1/2 \cdot 3^{1/2}$
2	5	7	$1/2$				

and zero for other permutations of the indices.

All this has been done in terms of the MATRIX REPRESENTATION of SU(3). A collection of *operators* F_i satisfying the commutation relations

$$[F_i, F_j] = 2i \sum_{k=1}^{k=8} f_{ijk} F_k$$

with the same structure constants f_{ijk} is also an SU(3) Lie algebra.

We now try to find a physical representation for the F_i

The matrices $\underline{\lambda}_1$, $\underline{\lambda}_2$ and $\underline{\lambda}_3$ are like the Pauli matrices, augmented by one row and one column.

As a result, F_1, F_2 and F_3 satisfy the SU(2) isospin relations — we can identify them with ISOSPIN.

The matrix $\underline{\lambda}_8$ commutes with each of the matrices $\underline{\lambda}_1$, $\underline{\lambda}_2$ and $\underline{\lambda}_3$. This means that in the corresponding OPERATOR picture the SU(3) basis states can be simultaneous eigenstates of F_8 and the isospin operators F_1, F_2, F_3.

Conventionally, the choice made is to assign F_8 to HYPERCHARGE

$$Y = S + B$$

For consistency of normalization, the scaled operator

$$F_8 = \frac{\sqrt{3}}{2} Y \equiv M$$

is a better choice than Y itself.

SHIFT OPERATORS AND WEIGHT DIAGRAMS

By analogy with the shift operators $J_\pm$ of $SU(2)$, we can define $SU(3)$ shift operators:-

$$I_\pm = F_1 \pm i F_2$$
$$U_\pm = F_6 \pm i F_7$$
$$V_\pm = F_4 \pm i F_5$$

In order to understand the rôle of these operators, we define the 2-component vector $\underline{G} = (I_3, M)$

5.

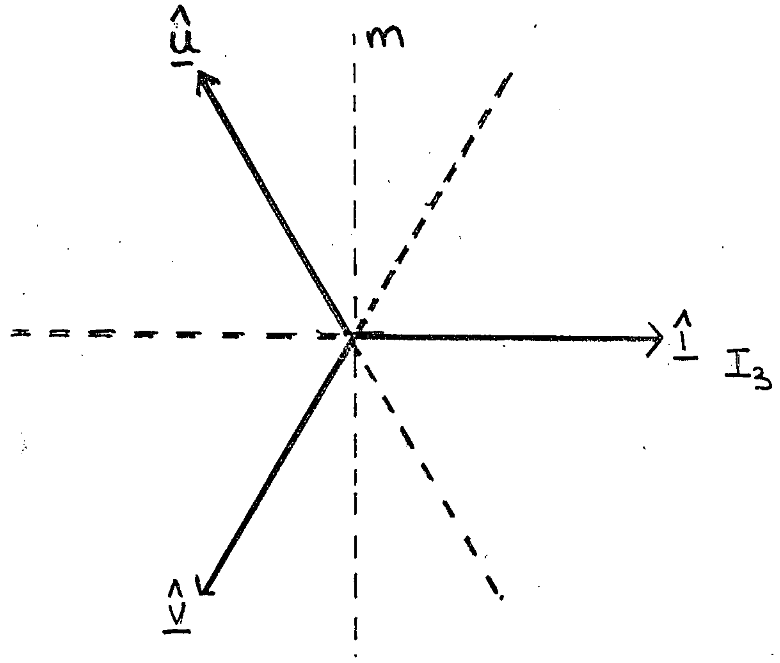
In this way, each eigenstate corresponds to a point $\underline{g} = (I_3, m)$ on a plane.

We now define directions in the $I_3 - m$ plane.

$$\hat{\underline{1}} = (1, 0)$$

$$\hat{\underline{u}} = \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$$

$$\hat{\underline{v}} = \left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right)$$



Using these, we find the following commutation relations:-

$$[\underline{G}, I_\pm] = \underline{\hat{\imath}}\, I_\pm \qquad [I_\pm, U_\pm] = \pm V_\mp$$

$$[I_+, I_-] = 2\underline{\hat{\imath}} . \underline{G} \qquad [U_\pm, V_\pm] = \pm I_\mp$$

$$[\underline{G}, U_\pm] = \pm \underline{\hat{u}}\, U_\pm \qquad [V_\pm, I_\pm] = \pm U_\mp$$

$$[U_+, U_-] = 2\underline{\hat{u}} . \underline{G}$$

$$[\underline{G}, V_\pm] = \pm \underline{\hat{v}}\, V_\pm$$

$$[V_+, V_-] = 2\underline{\hat{v}} . \underline{G}$$

N.B. the notation $[\underline{G}, I_\pm] = \underline{\hat{\imath}}\, I_\pm$ is shorthand for

$$[I_3, I_\pm] = \pm I_\pm \qquad [M, I_\pm] = 0.$$

WHY HYPERCHARGE?

A correspondence has been set up between the Gell-Mann matrices and operators representing isospin and hypercharge.

The hexagonal symmetry of I, U and V suggest we could also do the job in terms of "U-spin" or "V-spin". The operators "U_3" and "V_3" are obtained by a rotation in the I_3-M plane.

$$U_3 = -\tfrac{1}{2} I_3 + \frac{\sqrt{3}}{2} M$$

$$V_3 = -\tfrac{1}{2} I_3 - \frac{\sqrt{3}}{2} M$$

U-spin is interesting because the "other" operator corresponding to M is the electric charge, Q.

In terms of isospin and hypercharge

$$Q = I_3 + \tfrac{1}{2} Y = I_3 + \tfrac{1}{\sqrt{3}} M$$

The U-spin operators all commute with the charge operator:-

$$[U_\pm, Q] = 0 \ ; \ [U_3, Q] = 0.$$

1.

We have still not found the CASIMIR OPERATORS for SU(3)

Definition: The number of mutually commuting operators is the RANK of the group. It can be shown that the number of Casimir operators for the group equals the rank of the group. [H&M p.43]

The Casimir operator equivalent to J^2 for SU(3) is

$$F^2 = \sum_{i=1}^{i=8} F_i F_i$$

which commutes with each of the operators F_j.

o.

The other Casimir operator is of no physical interest. Its MATRIX representation is

$$G^3 = \sum_{ijk} \underline{A}_{ij} \underline{A}_{jk} \underline{A}_{ki}$$

where the MATRICES $\underline{A}_{ij}$ are given by

$$(A_{ij})_{kl} = \delta_{ik}\delta_{jl} - \tfrac{1}{3}\delta_{ij}\delta_{kl}$$

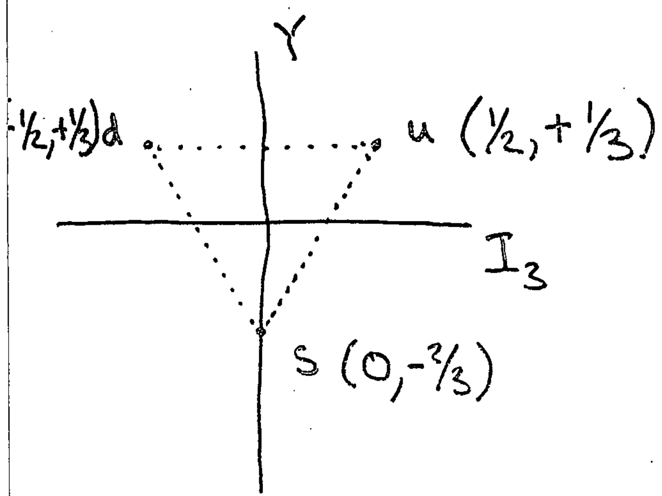
$$(i,j,k,l = 1,2,3)$$

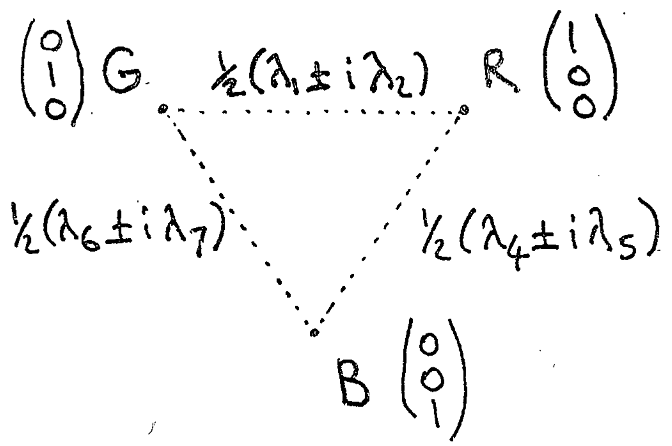
SU(3) Multiplets

All members of an SU(3) multiplet have the same value of the Casimir operator (as in SU(2)). The structure of SU(3) multiplets can be determined with WEIGHT DIAGRAMS and YOUNG TABLEAUX.

WEIGHT DIAGRAMS [Gibson & Pollard pp 268-282]	Method for finding extreme I_3, Y values of multiplet, then moving round it with shift operators.
YOUNG TABLEAUX [Close pp 47-53]	Graphical method of determining the decomposition of direct products of SU(3) multiplets.

Note that this discussion has been illustrated with operators from *flavour* SU(3). Analogous operators are used in QCD for *colour* SU(3).





QUARK WAVEFUNCTIONS

The quantum numbers for the observed hadrons can be arranged into SU(3) multiplets.

N.B. these are not the continuous groups of unitary operators but a set of basis vectors for an irreducible representation of the group.

IRREDUCIBLE REPRESENTATION of a Lie group. (working definition) A set of eigenstates of the GENERATORS such that any one is accessible from any other by successive application of shift operators.

Reminder of quark properties.

Flavour	u	d	s	c	b	t
Charge	$2/3$	$-1/3$	$-1/3$	$+2/3$	$-1/3$	$+2/3$
Isospin	$1/2$	$1/2$	0	0	0	0
I_3	$+1/2$	$-1/2$	0	0	0	0
Strangeness	0	0	-1	0	0	0
Charm	0	0	0	$+1$	0	0
Beauty	0	0	0	0	-1	0
Truth	0	0	0	0	0	$+1$
Baryon Number	$1/3$	$1/3$	$1/3$	$1/3$	$1/3$	$1/3$
Mass (MeV)	~ 1	~ 6	~ 400	~ 1.5 GeV	~ 5 GeV	173 GeV

We concentrate on the light quarks, and study ways of combining these to make hadrons.

Quarks	SYMMETRIC	ANTISYMMETRIC
$u\ u$	uu	
$u\ d$	$\frac{1}{\sqrt{2}}(uu+dd)$	$\frac{1}{\sqrt{2}}(uu-dd)$
$d\ d$	dd	

i.e. a TRIPLET which is SYMMETRIC under interchange of quark labels, and a SINGLET which is ANTISYMMETRIC.

Now consider combinations of THREE quarks (just u and d for now).

	$I_3 = 3/2$	$I_3 = +1/2$	$I_3 = -1/2$	$I_3 = -3/2$
Sym.	uuu	$\frac{1}{\sqrt{3}}(uud + udu + duu)$	$\frac{1}{\sqrt{3}}(udd + dud + ddu)$	ddd
Mixed (A)		$\frac{1}{\sqrt{2}}(ud-du)u$	$\frac{1}{\sqrt{2}}(ud-du)d$	
Mixed (S)		$-\frac{1}{\sqrt{3}}\left[\frac{(ud+du)u}{\sqrt{2}} - \sqrt{2}\,uud\right]$	$\frac{1}{\sqrt{3}}\left[\frac{(ud+du)d}{\sqrt{2}} - \sqrt{2}\,ddu\right]$	

The symmetric quadruplet is obtained from the (manifestly symmetric) uuu state by application of the shift operators

$$[\tau_- |u\rangle = |d\rangle]$$

Consider the ways the $I_3 = +\frac{1}{2}$ combinations can be made up.

$$(uu)d \qquad |1,1;\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle_S + \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle_S$$

$$\left(\frac{1}{\sqrt{2}}(ud+du)\right)u \qquad |1,0;\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle_S - \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle_S$$

$$\left(\frac{1}{\sqrt{2}}(ud-du)\right)u \qquad |0,0;\tfrac{1}{2}\,\tfrac{1}{2}\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_A \qquad [M_A]$$

Rearranging

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$[S]\ |\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\left\{uud + (ud+du)u\right\} = \sqrt{\tfrac{1}{3}}\,(uud+udu+duu)$$

$$[M_S]\ |\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\left\{\sqrt{2}\,uud - \frac{1}{\sqrt{2}}(ud+du)u\right\}$$

For the "MIXED SYMMETRY" states, interchange of particles leads (in general) to a linear superposition of M_S and M_A states.

EXAMPLE: Take $\sqrt{\frac{1}{3}}\left(\sqrt{2}\,u_1u_2d_3 - \frac{1}{\sqrt{2}}(u_1d_3+d_3u_1)u_2\right)$

swap $1 \leftrightarrow 3$

Get $$\sqrt{\tfrac{1}{3}}\left\{\sqrt{2}\,duu - \frac{(du+ud)u}{\sqrt{2}}\right\} = \sqrt{\tfrac{1}{6}}\,(du-ud)u$$

which is the M_A state.

QUARK WAVE FUNCTIONS (Cont.)

Having developed baryon wavefunctions for non-strange baryons, we now extend the formalism to include strange quarks.

We are combining 3 quarks of 3 flavours. These form 27 states, which decompose into multiplets as follows:-

$$3 \otimes 3 \otimes 3 = (10_S \oplus 8_{M,S}) \oplus (8_{M,A} + 1_A)$$

i.e. we get a decuplet, two octets of mixed symmetry, and an antisymmetric singlet.

The (symmetric) decuplet is made up of states of the form

$$\frac{1}{\sqrt{3}}\left(q_1 q_2 q_3 + q_2 q_3 q_1 + q_3 q_1 q_2\right)$$

with appropriate flavours.

2.

The totally antisymmetric state is

$$\Lambda_1^0 = \frac{1}{\sqrt{6}}(sdu - sud + usd - dsu + dus - uds)$$

The mixed symmetry states can be built up from the u and d states.

	M, S	M, A	
p	$\frac{1}{\sqrt{6}}[(ud+du)u - 2uud]$	$\frac{1}{\sqrt{2}}(ud-du)u$	
n	$-\frac{1}{\sqrt{6}}[(ud+du)d - 2ddu]$	$\frac{1}{\sqrt{2}}(ud-du)d$	
Σ^+	$\frac{1}{\sqrt{6}}[(us+su)u - 2uus]$	$\frac{1}{\sqrt{2}}(us-su)u$	$u_-\lvert d\rangle \Rightarrow \lvert s\rangle$ on p
Σ^-	$\frac{1}{\sqrt{6}}[(ds+sd)d - 2dds]$	$\frac{1}{\sqrt{2}}(ds-sd)d$	τ_- on Σ^0
Ξ^-	$-\frac{1}{\sqrt{6}}[(ds+sd)s - 2ssd]$	$\frac{1}{\sqrt{2}}(ds-sd)s$	τ_- on Ξ^0
Ξ^0	$-\frac{1}{\sqrt{6}}[(us+su)s - 2ssu]$	$\frac{1}{\sqrt{2}}(us-su)s$	u_- on Σ^0
Σ^0	$\frac{1}{\sqrt{12}}[s(du+ud) + (dsu+usd) - 2(du+ud)s]$	$\frac{1}{2}[(dsu+usd) - s(ud+du)]$	τ_- on Σ^+
Λ^0	$\frac{1}{\sqrt{2}}\left[\frac{dsu-usd}{\sqrt{2}} + \frac{s(du-ud)}{\sqrt{2}}\right]$	$\frac{1}{\sqrt{12}}[s(du-ud) + (usd-dsu) - 2(du-ud)s]$	orthog. Σ^0

3.

As quarks are FERMIONS, the overall quark wavefunctions should be ANTISYMMETRIC. While at present it is not clear how to handle the MIXED SYMMETRY flavour wavefunctions, remember these have to be combined with spatial, spin and colour wavefunctions to construct the overall wavefunction. As we shall see, the SPIN wavefunction (also mixed symmetry) can be combined with the flavour component to make a state of definite symmetry.

For COLOUR, it is supposed that the only allowed wavefunction is the (totally antisymmetric) triplet

$$\frac{1}{\sqrt{6}}(RGB - RBG + BRG - BGR + GBR - GRB)$$

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4. $\boxed{\text{MESONS}}$

Meson wavefunctions are formed from the direct product of a (q) TRIPLET and a ($\bar{q}$) ANTITRIPLET. These form an octet and a singlet.

$$3 \otimes \bar{3} = \underline{1} \oplus 8$$

Wavefunctions involving charge or strangeness are straightforward

e.g. $K^+ \rightarrow \bar{s}u$

However, SEVERAL states possible with $I_3 = 0, Y = 0$. There is a member of an isospin triplet

$$|\underline{8}, \underline{3}\rangle = \frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u}) \qquad \text{(SYMMETRIC)}$$

su(3) mult   su(2) mult.

and an asymmetric isospin singlet su(3) singlet

$$|\underline{1}, \underline{1}\rangle = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

There is a third state orthogonal to the other two

$$|\underline{8}, \underline{1}\rangle = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

5.

Notice that the correspondence between form and particle exchange symmetry is different for $(q\bar{q})$ mesons and (qq) diquarks.

cf

$\frac{1}{\sqrt{2}}(uu + dd)$ DIQUARK: SYMMETRIC

$\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$ MESON: ANTISYMMETRIC

Quarks and antiquarks behave DIFFERENTLY under rotations in isospin space.

Consider a rotation about the y-axis for a quark doublet

$$\phi = \begin{pmatrix} u \\ d \end{pmatrix}$$

The rotation operator is $\mathcal{U} = e^{-i\frac{\theta}{2}\underline{\sigma_2}}$

$$= 1 + i\frac{\theta}{2}\underline{\sigma_2} + \frac{1}{2!}\left(\frac{i\theta}{2}\right)^2\underline{\sigma_2}^2 + \frac{1}{3!}\left(\frac{i\theta}{2}\right)^3\underline{\sigma_2}^3 + \cdots$$

as

$$(\underline{\sigma_2})^{2n} = \underline{1}$$

so

$$\mathcal{U} = \left(1 + \frac{1}{2!}\left(\frac{i\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{i\theta}{2}\right)^4 + \ldots\right) + i\left(\frac{\theta}{2}\sigma_2 - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3\sigma_2 \ldots\right)$$

$$= \cos\frac{\theta}{2} + i\,\underline{\sigma_2}\sin\frac{\theta}{2}$$

6.

Thus

$$\left.\begin{aligned} u' &= \cos\frac{\theta}{2}\, u + \sin\frac{\theta}{2}\, d \\ d' &= -\sin\frac{\theta}{2}\, u + \cos\frac{\theta}{2}\, d \end{aligned}\right\} [A]$$

Now apply CHARGE CONJUGATION

$$u \rightarrow \bar{u}$$
$$d \rightarrow \bar{d}$$

$$\left.\begin{aligned} \bar{u}' &= \cos\frac{\theta}{2}\, \bar{u} + \sin\frac{\theta}{2}\, \bar{d} \\ \bar{d}' &= -\sin\frac{\theta}{2}\, \bar{u} + \cos\frac{\theta}{2}\, \bar{d} \end{aligned}\right\} [B]$$

The problem comes if isospin is taken into account. We can write [A] as

$$\begin{array}{l} I_3 = +\tfrac{1}{2} \\ I_3 = -\tfrac{1}{2} \end{array} \begin{pmatrix} u' \\ d' \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

For [B] we get

$$\begin{array}{l} I_3 = +\tfrac{1}{2} \\ I_3 = -\tfrac{1}{2} \end{array} \begin{pmatrix} \bar{d}' \\ \bar{u}' \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \bar{d} \\ \bar{u} \end{pmatrix}$$

so this definition of the antiquark doublet leads to different rotation properties in isospin space.

7. We recover the SAME rotation properties if we define the antiquark doublet corresponding to $\binom{u}{d}$ to be $\binom{\bar{d}}{-\bar{u}}$

$$\begin{pmatrix} \bar{d}' \\ -\bar{u}' \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$$

Thus the equivalent of $\frac{1}{\sqrt{2}}(uu + dd)$ is $\frac{1}{\sqrt{2}}(d\bar{d} + u(-\bar{u})) = \frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$

This discussion arises from the deeper problem of establishing a phase convention for isospin states when antiparticles are considered.

The charge conjugation operator U_c changes the sign of I_3.

In operator formalism

$$I_3^c = U_c I_3 U_c^\dagger = -I_3$$

8.

Similarly for the shift operators

$$I_{\pm}^{c} = U_c I_{\pm} U_c^{\dagger}$$

so we can transform

$$I_3 I_{\pm} - I_{\pm} I_3 = \pm I_{\pm}$$

$$I_3 U_c^{\dagger} U_c I_{\pm} - I_{\pm} U_c^{\dagger} U_c I_3 = \pm I_{\pm}$$

$$U_c I_3 U_c^{\dagger} U_c I_{\pm} U_c^{\dagger} - U_c I_{\pm} U_c^{\dagger} U_c I_3 U_c^{\dagger} = \pm U_c I_{\pm} U_c^{\dagger}$$

$$[I_3^c \, I_{\pm}^c] = \pm I_{\pm}^c$$

Now replace I_3^c by I_3

$$[I_3 , I_{\pm}^c] = \mp I_{\pm}^c$$

This is satisfied BOTH if

$$I_{\pm}^c = - I_{\mp}$$

CHECK: $[I_3, -I_{\mp}] = \mp - I_{\mp} = \pm I_{\mp}$

$$-I_3 I_- + I_- I_3 = + I_- \Rightarrow I_- I_3 - I_3 I_- = I_- \quad \text{OK}$$

$$-I_3 I_+ + I_+ I_3 = - I_+ \Rightarrow I_+ I_3 - I_3 I_+ = - I_+$$

$$I_3 I_+ - I_+ I_3 = I_+ \quad \text{OK}$$

AND if

$$I_{\pm}^{c} = +I_{\mp} \qquad (!!)$$

CHECK: $[I_3, I_\mp] = \mp I_\mp$

$$I_3 I_- - I_- I_3 = -I_- \qquad \text{OK}$$
$$I_3 I_+ - I_+ I_3 = +I_+ \qquad \text{OK}$$

We can summarize this as

$$I_{\pm}^{c} = \alpha I_{\mp} \qquad (\alpha = \pm 1)$$

But this means (e.g.) that

$$I_{+}^{c} = I_1^c + i I_2^c = \alpha I_- = \alpha(I_1 - i I_2)$$

so
$$I_1^c = \alpha I_1$$
$$I_2^c = -\alpha I_2$$

i.e. WHICHEVER choice is made for α, I_1, I_2, I_3 do not transform the same way under U_c.

So great care is required in following phase conventions.

CLOSE [An Introduction to Quarks and Partons, Academic Press, 1980] requires

- quarks transform into antiquarks with POSITIVE relative signs

$$U_c|u\rangle = |\bar{u}\rangle$$
$$U_c|d\rangle = |\bar{d}\rangle$$
$$U_c|s\rangle = |\bar{s}\rangle$$

- The shift operators τ_- and u_- have POSITIVE coefficients

$$\tau_-|u\rangle \Rightarrow |d\rangle$$

$$u_-|d\rangle \Rightarrow |s\rangle$$

The tables of quark flavour wavefunctions shown earlier use these conventions.

G PARITY STATES FOR MESONS

Any given meson state (e.g. $u\bar{d}$)
we can make symmetric and antisymmetric combinations

$$\frac{1}{\sqrt{2}}\left(u\bar{d} + \bar{d}u\right) \qquad \text{SYMMETRIC}$$

$$\frac{1}{\sqrt{2}}\left(u\bar{d} - \bar{d}u\right) \qquad \text{ANTISYMMETRIC}$$

These have different properties under G-parity.

The G-parity operator consists of the operation U_c followed by a rotation π about the I_2 axis.

We examine its properties in the 2×2 matrix representation.

The rotation is represented by the matrix $e^{-i\pi\sigma_2} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Noting signs in the antiquark doublet,

we represent

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_q, \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_q, \quad \bar{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{\bar{q}}, \quad \bar{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\bar{q}}$$

Then (e.g.)

$$Gd \Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\{u_c d\} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\{\bar{d}\} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\bar{q}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\bar{q}} = -\bar{u}$$

In this way we obtain

$$G u = \bar{d}$$
$$G d = -\bar{u}$$
$$G \bar{u} = d$$
$$G \bar{d} = -\bar{u}$$

Now examine the G-parity of the $q\bar{q}$ symmetric and antisymmetric states.

$$G\left\{\frac{1}{\sqrt{2}}(u\bar{d} + \bar{d}u)\right\} = \frac{1}{\sqrt{2}}\left\{-\bar{d}u - u\bar{d}\right\} = -\left\{\frac{1}{\sqrt{2}}(u\bar{d} + \bar{d}u)\right\}$$

$$G\left\{\frac{1}{\sqrt{2}}(u\bar{d} - \bar{d}u)\right\} = \frac{1}{\sqrt{2}}\left\{-\bar{d}u + u\bar{d}\right\} = +\left\{\frac{1}{\sqrt{2}}(u\bar{d} - \bar{d}u)\right\}$$

13.

STRANGE quarks are ISOSINGLETS so they are invariant under isospin rotations.

$$U_G \, s = \bar{s}$$

$$U_G \, \bar{s} = s$$

We can thus examine G-Parity for Kaons:-

$$U_G K^+ = U_G \left\{ \frac{1}{\sqrt{2}} (u\bar{s} + \bar{s}u) \right\} = \left\{ \frac{1}{\sqrt{2}} (\bar{d}s + s\bar{d}) \right\}$$

$$= \overline{K^0}$$

①

WAVEFUNCTIONS WITH SPIN

In order to specify the hadron wavefunction completely, it is necessary to define the spin wavefunction. Quarks are spin-½ fermions and can therefore take values up (↑) and down (↓)

Therefore, the combinations one can make are the analogues of the ud flavou wavefunctions studied earlier. For MESONS

	$S=1$	$S=0$
$S_3=1$	$\uparrow\uparrow$	
$S_3=0$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)$
$S_3=-1$	$\downarrow\downarrow$	
	Symmetric χ_S	Antisymmetric χ_A

(2) These can now be combined with the symmetric and antisymmetric *flavour* wavefunctions to obtain combinations of definite exchange symmetry.

SYMMETRIC	$\phi_S \chi_S$	$\phi_A \chi_A$
ANTISYMMETRIC	$\phi_S \chi_A$	$\phi_A \chi_S$

We now compare these combinations with the quantum numbers for the important 0^{-+} and 1^{--} mesons. For the $I_3 = 0$ members:-

0^{-+}	π^0	$C=+$	$S=0$
1^{--}	ρ^0	$C=-$	$S=1$

C-parity fixes $\phi_{S,A}$ to be ϕ_S (ϕ_A) for 0^- (1^-) while spin determines $\chi_{S,A}$ to be χ_A (χ_S) for 0^- (1^-). Thus BOTH combinations are overall ANTISYMMETRIC.

③

Note that this means that the colour wavefunction must be

- symmetric under particle interchange
- an SU(3) colour SINGLET.

as overall a two-fermion wavefunction must be ANTISYMMETRIC.

BARYONS

Baryons are treated in an analogous way to mesons – made more complicated by the mixed symmetry wavefunctions.

For the spin part, we obtain

	$S = 3/2$	$S = 1/2$	$S = 1/2$
$S_3 = +3/2$	$\uparrow\uparrow\uparrow$		
$S_3 = +1/2$	$\frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$	$\frac{1}{\sqrt{6}}[(\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow]$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow$
$S_3 = -1/2$	$\frac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)$	$-\frac{1}{\sqrt{6}}[(\uparrow\downarrow + \downarrow\uparrow)\downarrow - 2\downarrow\downarrow\uparrow]$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\downarrow$
$S_3 = -3/2$	$\downarrow\downarrow\downarrow$		
	S	M_S	M_A

Note that this is the same structure as was developed for the ud flavour wavefunction

If we now consider all combinations of spin and flavour wavefunctions, we get the following exchange symmetry properties:-

Spin \ Flavour	S	M	A
S	S	M	A
M	M	S, M, A	M

⑤

Considering u, d and s quarks only we obtain 6 basic states

$$u\uparrow \quad u\downarrow \quad d\uparrow \quad d\downarrow \quad s\uparrow \quad s\downarrow$$

plus a similar number for antiquarks. It turns out that combinations of these states are well-described by the SU(6) group. This time we do not derive the matrix representation but note that it comes from combining $SU(3)_{flavour}$ with $SU(2)_{spin}$.

MULTIPLET STRUCTURE

For mesons the SU(6) decomposition gives

$$\underline{6} \otimes \underline{\overline{6}} = \underset{\uparrow \atop \phi_A \chi_A}{\underline{1}} \oplus \underline{35}$$

$$\underline{35} \longrightarrow (\underline{8},\underline{1}) \oplus (\underline{1},\underline{3}) + (\underline{8},\underline{3})$$

⑥ For baryons we have

$$\underline{6} \otimes \underline{6} \otimes \underline{6} = \underline{56}_S \oplus \underline{70}_{M,S} \oplus \underline{70}_{M,A} + \underline{20}_A$$

The multiplets can be further expanded as follows:-

S: $\underline{56} = (\underline{10}, \underline{4}) \oplus (\underline{8}, \underline{2})$

M: $\underline{70} = (\underline{10}, \underline{2}) \oplus (\underline{8}, \underline{4}) \oplus (\underline{8}, \underline{2}) \oplus (\underline{1}, \underline{2})$

A: $\underline{20} = (\underline{8}, \underline{2}) \oplus (\underline{1}, \underline{4})$

[N.B. $(\underline{10}, \underline{4})$ means an SU(3) decuplet with spin-3/2 (giving a (2s+1)-plet in spin); $(\underline{8}, \underline{2})$ is an SU(3) octet with spin-1/2, etc.]

The full decomposition of the quark wavefunction is the direct product of the spin and flavour parts.

For a Δ^{++} with $S_3 = 1/2$

$$uuu \otimes \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$

$$= \frac{1}{\sqrt{3}}(u\uparrow u\uparrow u\downarrow + u\uparrow u\downarrow u\uparrow + u\downarrow u\uparrow u\uparrow)$$

⑦ The physical $\frac{1}{2}^+$ baryons are a linear superposition of mixed symmetry states giving an overall symmetric combination

$$(\underline{10}, \underline{4}) = \phi_S \chi_S$$

$$(\underline{8}, \underline{2}) = \frac{1}{\sqrt{2}}\left(\phi_{M,S}\chi_{M,S} + \phi_{M,A}\chi_{M,A}\right)$$

As an example, we expand the $S_3 = +\frac{1}{2}$ wavefunction for a Λ^0

I $\quad \phi_{M,S}\chi_{M,S}$

$$\frac{1}{2}\{dsu - usd + sdu - sud\} \cdot \frac{1}{\sqrt{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow)$$

$$\begin{aligned}
= {} & \cancel{d\uparrow s\downarrow u\uparrow}^{(i)} - \cancel{u\uparrow s\downarrow d\uparrow}^{(ii)} + s\uparrow d\downarrow u\downarrow - s\uparrow u\downarrow d\uparrow \\
& + d\uparrow s\uparrow u\uparrow - u\downarrow s\uparrow d\uparrow + \cancel{s\downarrow d\uparrow u\uparrow}^{(iii)} - \cancel{s\downarrow u\uparrow d\uparrow}^{(iv)} \\
& - 2d\uparrow s\uparrow u\downarrow - 2u\uparrow s\uparrow d\downarrow + 2s\uparrow d\uparrow u\downarrow - 2s\uparrow u\uparrow d\downarrow
\end{aligned}$$

II $\quad \phi_{M,A}\chi_{MA}$

$$\frac{1}{\sqrt{6}}\left(sdu - sud + usd - dsu - 2(dus - uds)\right) \cdot \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow$$

$$\begin{aligned}
& s\uparrow d\downarrow u\uparrow - s\uparrow u\downarrow d\uparrow + \cancel{u\uparrow s\downarrow d\uparrow}^{(ii)} - \cancel{d\uparrow s\downarrow u\uparrow}^{(i)} - 2d\uparrow u\downarrow s\uparrow + 2u\uparrow d\downarrow s\uparrow \\
& - \cancel{s\downarrow d\uparrow u\uparrow}^{(iii)} + \cancel{s\downarrow u\uparrow d\uparrow}^{(iv)} - u\downarrow s\uparrow d\uparrow + d\downarrow s\uparrow u\uparrow - 2d\downarrow u\uparrow s\uparrow - 2u\downarrow d\uparrow s\uparrow
\end{aligned}$$

$$= s\uparrow\{(d\downarrow u\uparrow + d\uparrow u\downarrow) - (u\downarrow d\uparrow + u\uparrow d\downarrow)\}$$

$$+ \{(u\uparrow d\downarrow - u\downarrow d\uparrow) + (d\downarrow u\uparrow - d\uparrow u\downarrow)\} s\uparrow$$

$$+ \underline{(d\downarrow s\uparrow u\uparrow - d\uparrow s\uparrow u\downarrow) - (u\downarrow s\uparrow d\uparrow + u\uparrow s\uparrow d\downarrow)}$$

Notice that the u and d quarks are in an $S_3 = 0$ combination, and the spin of the Λ is carried by the s quark.

This property is often used in studies of s-quark polarization.

QUARK MODEL APPLICATIONS

CHARGE. (*Flavour only*)

Consider a NUCLEON which is a spin ½ baryon (so a member of the $\frac{1}{2}^+$ OCTET)

Its wavefunction is made up of M_S and M_A parts:-

$$\frac{1}{\sqrt{2}}(\phi_{MS}\chi_{MS} + \phi_{MA}\chi_{MA})$$

(9)

The nucleon CHARGE is obtained as the sum of matrix elements for charge operators acting on each of the three quarks.

$$Q = \sum_{i=1}^{i=3} \frac{1}{\sqrt{2}} \langle \phi_{M,S}\chi_{M,S} + \phi_{M,A}\chi_{M,A} | e_i | \phi_{M,S}\chi_{M,S} + \phi_{M,A}\chi_{M,A} \rangle \frac{1}{\sqrt{2}}$$

The choice of the third quark is ARBITRARY, so we can simplify this expression, as

$$\sum_{i=1}^{i=3} \langle \; | e_i | \; \rangle = 3 \langle \; | e_3 | \; \rangle$$

Remembering that the charge operator acts on the *flavour* wavefunction only, we can integrate the spin wavefunctions.

$$\langle \chi_{MS} | \chi_{MS} \rangle = \langle \chi_{MA} | \chi_{MA} \rangle = 1, \quad \langle \chi_{MS} | \chi_{MA} \rangle = 0$$

So

$$Q = \frac{3}{2} \left\{ \langle \phi_{MS} | e_3 | \phi_{MS} \rangle + \langle \phi_{MA} | e_3 | \phi_{MA} \rangle \right\}$$

(10)

For a PROTON

$\langle \phi_{MS} | e_3 | \phi_{MS} \rangle$

$= \frac{1}{6} \langle udu + duu - 2uud | e_3 | udu + duu - 2uud \rangle$

Since e_3 acts on 3rd quark only

$= \frac{1}{6} \left(\frac{2}{3} + \frac{2}{3} - \frac{4}{3} \right) = 0$

$\langle \phi_{MA} | e_3 | \phi_{MA} \rangle$

$= \frac{1}{2} \langle udu - duu | e_3 | udu - duu \rangle$

$= \frac{1}{2} \left(\frac{2}{3} + \frac{2}{3} \right) = \frac{2}{3}$

Thus for the CHARGE

$Q = \frac{3}{2} \left(0 + \frac{2}{3} \right) = 1$

A similar calculation for the neutron gives $Q = 0$ as expected.

STATIC MAGNETIC MOMENTS

(Flavour & Spin)

The MAGNETIC MOMENT for a baryon is given by

$$\sum_{i=1}^{i=3} \mu e_i (\sigma_3)_i \qquad \left(\text{i.e. } \sum_{i=1}^{i=3} \mu e_i (\sigma_z)_i \right)$$

As before we can use the third quark to obtain an expression

$$3 \mu e_3 (\sigma_z)_3$$

The subscript "3" implies the operator acts on the THIRD quark only.

For SPIN

$$\langle \chi_{MS} \uparrow | \sigma_z^{(3)} | \chi_{MS} \uparrow \rangle$$

$$= \frac{1}{6} \langle \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow | \sigma_z^{(3)} | \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow \rangle$$

$$= \frac{1}{6}(1+1-4) = -\frac{1}{3}$$

- $\langle \chi_{MA}\uparrow | \sigma_z^{(3)} | \chi_{MA}\uparrow \rangle$

$= \frac{1}{2} \langle \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow | \sigma_z^{(3)} | \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow \rangle$

$= \frac{1}{2}(1+1) = 1$

- $\langle \chi_{MS}\uparrow | \sigma_z^{(3)} | \chi_{MA}\uparrow \rangle$

$= \frac{1}{\sqrt{12}} \langle \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow | \sigma_z^{(3)} | \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow \rangle$

$= \frac{1}{\sqrt{2}}(1-1) = 0$

Combining these results with the FLAVOUR matrix elements we have already obtained, for the PROTON we have

$$\frac{3\mu}{2} \langle \phi_{MS}\chi_{MS}\uparrow + \phi_{MA}\chi_{MA}\uparrow | e_3 \sigma_z^{(3)} | \phi_{MS}\chi_{MS}\uparrow + \phi_{MA}\chi_{MA}\uparrow \rangle$$

$$= \frac{3\mu}{2}\left\{ -\frac{1}{3}\langle \phi_{MS}|e_3|\phi_{MS}\rangle + (1)\langle \phi_{MA}|e_3|\phi_{MA}\rangle \right\} = \frac{3}{2}\mu\left(0+\frac{2}{3}\right)$$
$$= \mu$$

For the NEUTRON

$$\frac{3\mu}{2} \langle \phi_{MS}\chi_{MS}\uparrow + \phi_{MA}\chi_{MA}\uparrow | e_3 \sigma_z^{(3)} | \phi_{MS}\chi_{MS}\uparrow + \phi_{MA}\chi_{MA}\uparrow \rangle$$

$$= \frac{3}{2}\mu\left\{ -\frac{1}{3}\langle \phi_{MS}|e_3|\phi_{MS}\rangle + (1)\langle \phi_{MA}|e_3|\phi_{MA}\rangle \right\} = \frac{3\mu}{2}\left(-\frac{1}{3}\times\frac{1}{3} + \left(-\frac{1}{3}\right)\right)$$
$$= -\frac{3\mu}{2}\left(\frac{4}{9}\right) = -\mu\frac{2}{3}$$

(13)

Thus $\frac{\mu_p}{\mu_n} = -\frac{3}{2}$

Experimentally $\mu_p/\mu_n = \frac{-2.79}{1.91} = -1.46$

For the light quarks, we have assumed quark magnetic moments

$$\mu_u = \mu_d = \mu$$

For heavier quarks, the quark mass must be taken into account.

HALZEN & MARTIN (pp 55, 107) obtain

$$\mu_i = Q_i \left(e/2m_i \right)$$

Q_i – scaling factor
m_i – quark mass

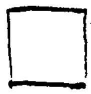
for Dirac STRUCTURELESS quarks. CLOSE uses $\mu_s \simeq 3/5\, \mu_d$. We expect $\mu_c \ll \mu_{s,u,d}$

YOUNG TABLEAUX

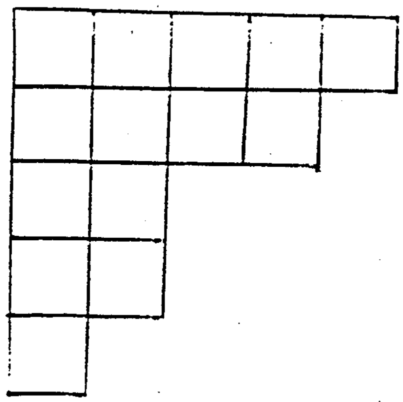
Young Tableaux provide a useful graphical technique for studying SU(N) multiplets. Through their use it is possible to

- Represent SU(N) multiplets,
- Display their particle interchange symmetries, and
- Develop a Clebsch-Gordon series, i.e. a decomposition of a direct product of two SU(N) multiplets into its irreducible representations.

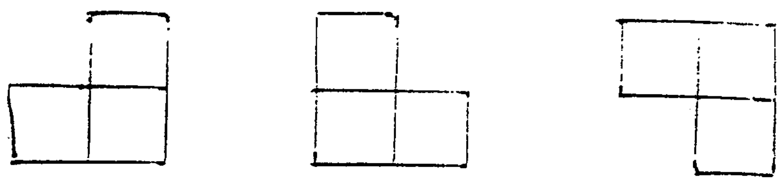
Each multiplet is represented by a pattern of boxes, called a tableau. In SU(N), the "basic representation" is an N-plet, represented by a single box.



More complicated multiplets are represente[d] by patterns of boxes like



The tableaux are aligned on the left. Each row must contain no more boxes than the row above, so figures like



are not legal. For SU(N) the number of column[s] must not exceed N.

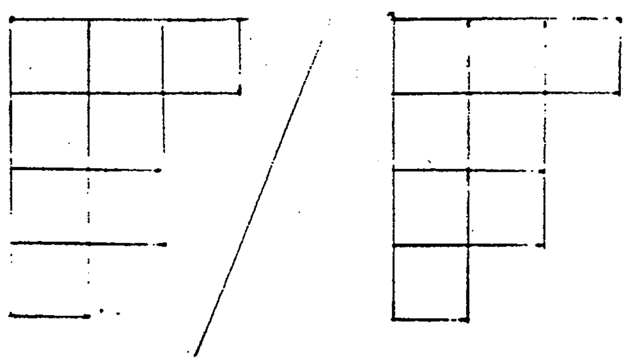
A ROW of boxes 

represents a multiplet completely *symmetric* under interchange of labels, while a COLUMN represents a totally *antisymmetric* multiplet. A column of N boxes can only exist for SU(N).

Tableaux which are neither rows nor columns have mixed interchange symmetry.

MULTIPLICITIES

The dimension to be associated with a given tableau is found by taking a "formal quotient" of two copies of the tableau:-



For the NUMERATOR, in SU(N), one assigns numbers to the boxes in the tableau as follows:

- The top left-hand box is assigned the number N.
- Moving rightwards along the top row, assign numbers N+1, N+2, N+3, ... to each successive box
- Assign N−1 to the leftmost box in the next row down, and ascending numbers N, N+1, ... to each box moving rightwards along the raw
- Repeat for the next row, starting with N−2 etc.

Thus the tableau shown in the example becomes

N	$N+1$	$N+2$
$N-1$	N	
$N-2$	$N-1$	
$N-3$		

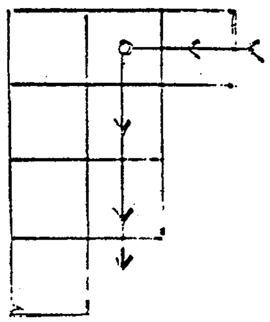
The DENOMINATOR is determined by assigning to each box

[the number of boxes IN THE SAME ROW to the right of the given box

+ [the number of boxes IN THE SAME COLUMN below the given box

+ 1

Note that this amounts to drawing a line starting at the right-hand end of the given row, and turning downwards at the box under consideration. The number obtained is called the "hook" for the given box



4 boxes traversed,

∴ hook = 4.

For the tableau shown, one obtains

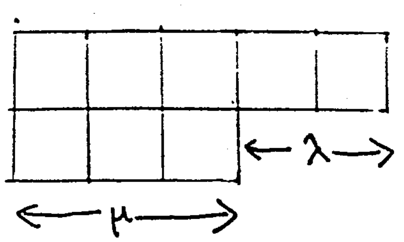
6	4	1
4	2	
3	1	
1		

One now obtains the product of all the numbers in the numerator and denominator tableaux, and divides the product. Thus for SU(6) the dimension of the tableau we have been using as an example is

$$\begin{array}{|c|c|c|}\hline \cancel{6} & 7 & 8 \\ \hline 5 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} \cancel{4} & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \cancel{3} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \Bigg/ \begin{array}{|c|c|c|}\hline \cancel{6} & 4 & 1 \\ \hline \cancel{4} & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \cancel{3} & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} = \frac{5 \times 5 \times 6 \times 7 \times 8}{2 \times 4} = 1050$$

so the tableau represent a 1050-plet in SU(6)

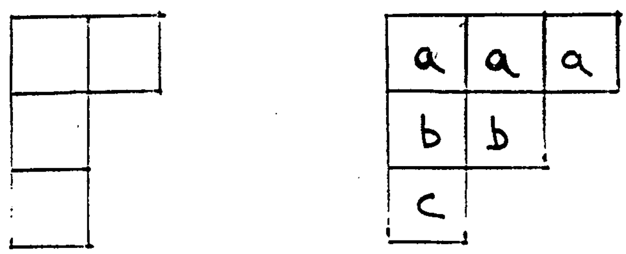
Specializing to SU(3), we find that tableaux used for SU(3) have no more than two columns. A tableau can be related to a weight diagram represented by (λ, μ) as follows:-



Clebsch-Gordon Series

One of the principal uses of Young Tableaux in particle physics is to determine the multiplet structure for direct products of SU(N) multiplets. The rules for evaluating a product, say $T_1 \times T_2$, are as follows:-

a.) Write down the two tableaux, T_1 and T_2, labelling successive rows of T_2 with indices a, b, c, ...



b.) Add boxes $\boxed{a} \ldots\ldots \boxed{a}$, $\boxed{b} \ldots\ldots \boxed{b}$, ... from T_2 to T_1, subject to the rules

(i) At each stage, the augmented T_1 diagram must be a legal Young Tablea-

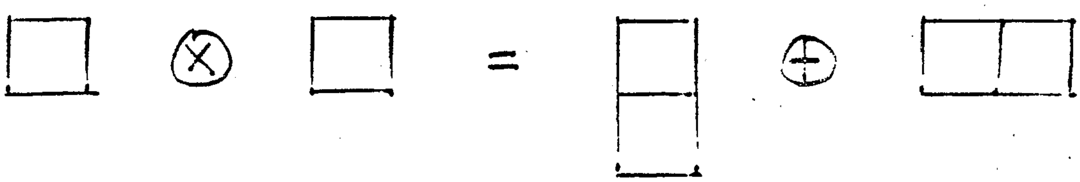
(ii) Boxes with the same label must not appear in the same column.

(iii) For a given box position, define n_a to be the number of a's ABOVE and TO THE RIGHT of it, similarly for n_b etc. Then $n_a \geqslant n_b \geqslant n_c$.

(c) If two tableaux of the same shape are produced, they are different only if the labels are differently distributed.

d) Cancel columns with N boxes.

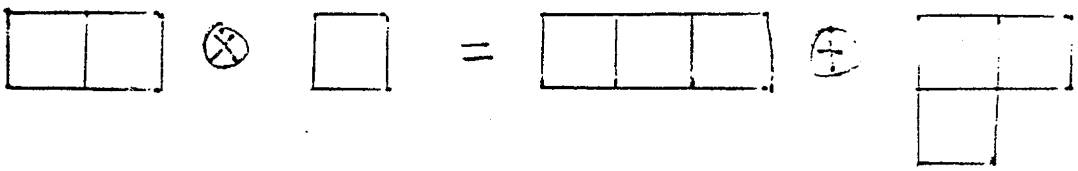
Simple examples of this are



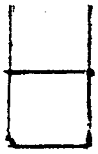
which means $3 \otimes 3 = \bar{3} \oplus 6$ in $SU(3)$, but

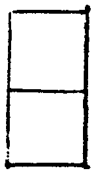
$6 \otimes 6 = 21 \oplus 15$ in $SU(6)$,

and

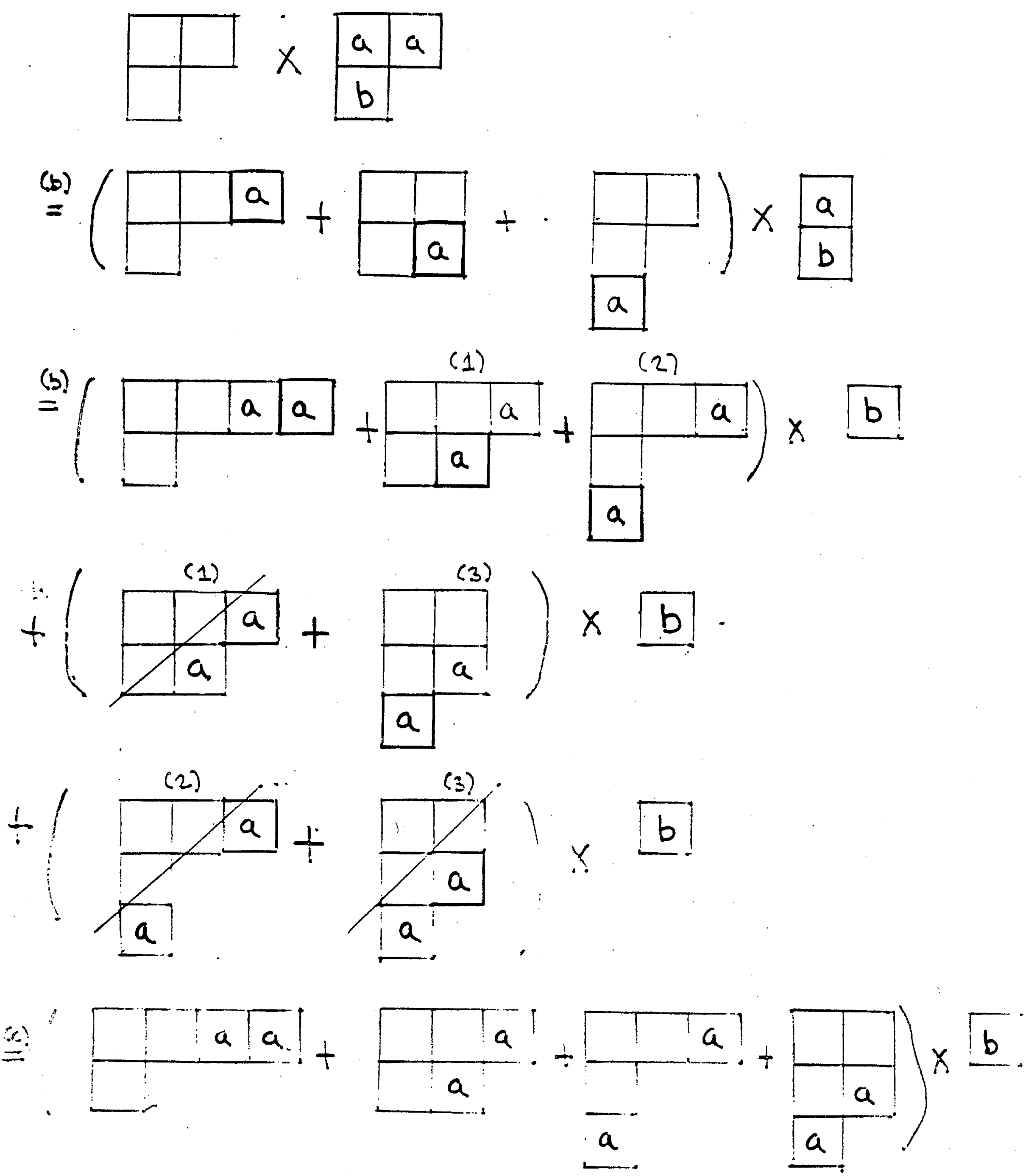


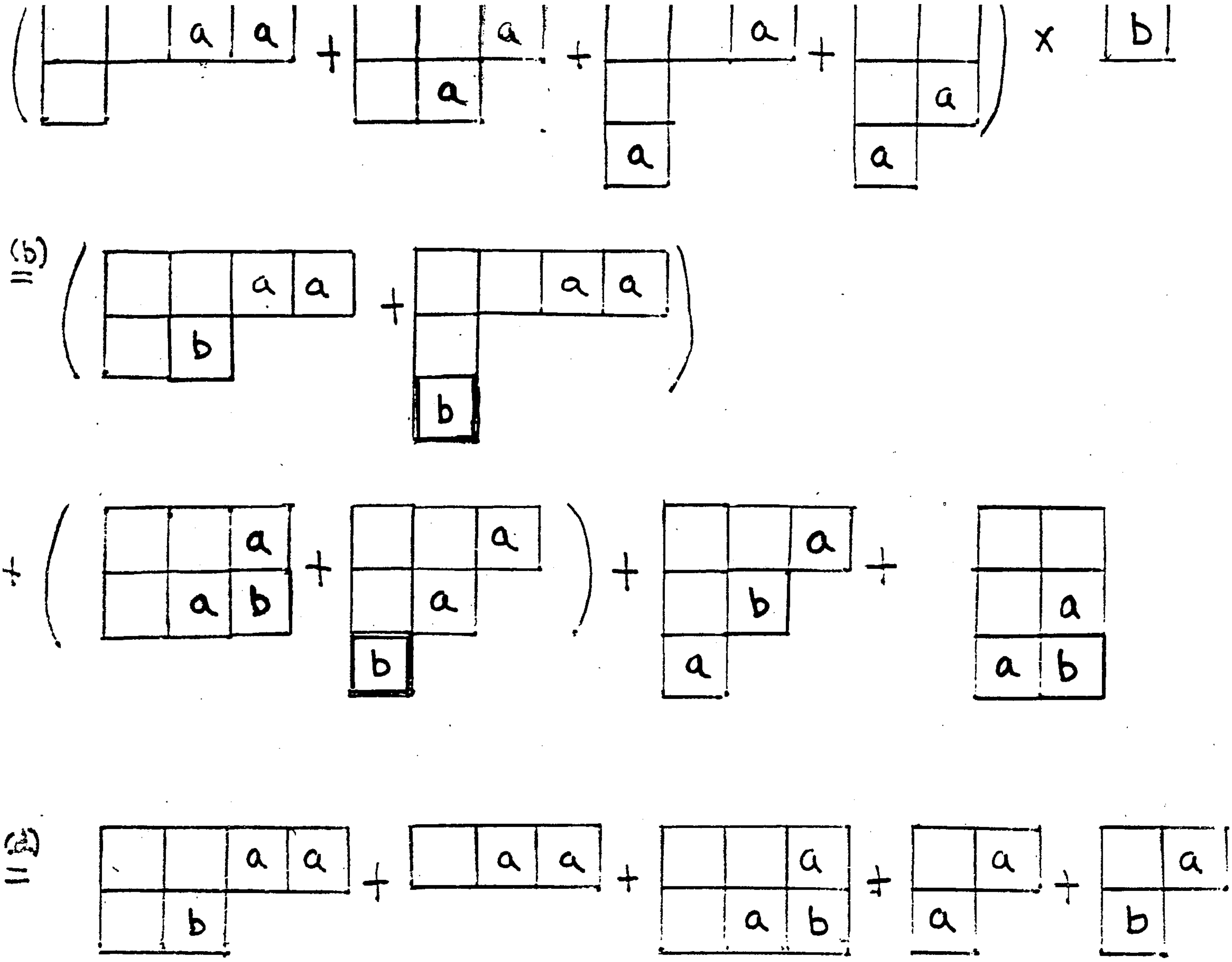
i.e. $\underline{6} \otimes \underline{3} = \underline{10} \oplus \underline{8}$

Note that  is a $\underline{3}$ with an inverted order of weights, and is written $\overline{3}$ or 3^*.

Note that a column vector of the type  (with N-1 columns for SU(N)) also represents a basic multiplet of charge-conjugate particles, e.g. a triplet of antiquarks instead of a triplet of quarks.

A more complicated example:-





$+ \ \underline{1}$

su(3)

$8 \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8 \oplus 8 \oplus 1$

①

MASS SPLITTINGS AND MIXING ANGLES

If SU(3) of FLAVOUR were an exact symmetry, all the hadrons in a given SU(3) multiplet would have the same mass.

Mass splittings within an SU(2) multiplet are small and basically electromagnetic in origin.

Much larger mass splittings occur at the SU(3) level; these reflect the mass differences between the different quark flavours.

(2) In the $3/2^+$ decuplet one sees a mass splitting of around 150 MeV when a u or d quark is replaced by an s quark.

$$\underbrace{\Delta(1236) - \Sigma^*(1385)}_{\Delta m = 149\,\text{MeV}} - \underbrace{\Xi^*(1530)}_{\Delta m = 145\,\text{MeV}} - \underbrace{\Omega^-(1670)}_{\Delta m = 140\,\text{MeV}}$$

If all multiplets were made up of pure SU(3) states, calculation of mass shifts would be straightforward (as it is for the $3/2^+$ baryons).. Most of the complications arise because of MIXING of pure SU(3) states in the physical states.

For example, in the meson sector, the OCTET ISOSINGLET and SU(3) SINGLET

③

$|\underline{8}, I = I_3 = Y = 0\rangle$ and $|\underline{1}, I = I_3 = Y = 0\rangle$

mix SUBSTANTIALLY except in the 0^- nonet.

$\boxed{\text{GELL-MANN MODEL}}$

We follow the original phenomenological mass shift model of Gell-Mann, with a few comments about the quark model interpretation.

We consider the strong interaction as consisting of two parts:-

- An SU(3) invariant "very strong" interaction
- A "medium strong" interaction

④

We ignore mass differences between members of an isospin multiplet.

To obtain simple transformation properties, Gell-Mann proposed that the symmetry-breaking term H_{MS} should be an SU(3) generator.

If members of a given isospin multiplet are to be regarded as degenerate states, this operator should commute with I_3; the obvious choice is the *hypercharge* operator, Y.

⑤

The members of a U-spin multiplet will display the mass-breaking effect.

However, the $|U=1, U_3=0\rangle$ state Σ^0_u is a superposition of the physical Σ^0 and Λ^0 states. To study the $(\frac{1}{2}^+)$ baryon sector, we must first determine the mixing.

Define

$$|\Sigma^0_u\rangle = \alpha|\Sigma^0\rangle + \beta|\Lambda\rangle$$

and note that

$$U_-|n\rangle = 2^{1/2}|\Sigma^0_u\rangle$$

$$I_-|\Sigma^+\rangle = 2^{1/2}|\Sigma^0\rangle$$

Using the adjoint form of the isospin equation

$$\langle\Sigma^+|I_-^\dagger = 2^{1/2}\langle\Sigma^0|$$

(6)

We write

$$2\langle\Sigma^0|\Sigma^0_u\rangle = 2\alpha = \langle\Sigma^+|I_-^\dagger U_-|n\rangle$$
$$= \langle\Sigma^+|I_+ U_-|n\rangle$$

Now, $[I_+, U_-] = 0$, so

$$= \langle\Sigma^+|U_- I_+|n\rangle$$
$$= \langle p|p\rangle$$
$$= 1$$

Thus

$$|\Sigma^0_u\rangle = \tfrac{1}{2}|\Sigma^0\rangle + \frac{\sqrt{3}}{2}|\Lambda^0\rangle$$

because $\alpha^2 + \beta^2 = 1$

The mass-breaking operator transforms as Y. In terms of U_3 and Q

$$Y = U_3 + \tfrac{1}{2}Q$$

i.e. Y transforms as a sum of a U-spin VECTOR and a U-spin SCALAR.

7

i.e.

$$H_{MS} = H_{MS}^{(v)} + H_{MS}^{(s)}$$

Using first-order perturbation theory, the shift to a system where all masses are equal is

$$\delta m_i = \langle i | H_{MS} | i \rangle$$

Consider the U-spin triplet

	n	Σ_u^0	Ξ^0
U_3	$+1$	0	-1

For the VECTOR component, we have

$$\langle U=1, U_3 | H_{MS}^{(v)} | U=1, U_3 \rangle = b\, U_3$$

and for the SCALAR component

$$\langle U=1, U_3 | H_{MS}^{(s)} | U=1, U_3 \rangle = a$$

Thus

$$\langle n | H_{MS} | n \rangle = a + b$$

$$\langle \Xi^0 | H_{MS} | \Xi^0 \rangle = a - b$$

and

$$\langle \Sigma_u^0 | H_{MS} | \Sigma_u^0 \rangle = \langle \tfrac{1}{2}\Sigma^0 + \tfrac{\sqrt{3}}{2}\Lambda^0 | H_{MS} | \tfrac{1}{2}\Sigma^0 + \tfrac{\sqrt{3}}{2}\Lambda^0 \rangle = a$$

So

$$\delta n = a + b$$

$$\delta \Xi^0 = a - b$$

$$\tfrac{1}{4}\delta\Sigma^0 + \tfrac{3}{4}\delta\Lambda = a$$

Eliminating a and b

$$\tfrac{1}{2}(\delta n + \delta \Xi^0) = \tfrac{1}{4}\delta\Sigma^0 + \tfrac{3}{4}\delta\Lambda^0$$

The mass differences are all taken with respect to a primordial mass m_0 (e.g. $m_n = m_0 + \delta_n$, $m_{\Sigma^0} = m_0 + \delta\Sigma^0$, etc.)

We can therefore replace δn by n etc. to obtain

$$\tfrac{1}{2}(n + \Xi^0) = \tfrac{1}{4}\Sigma^0 + \tfrac{3}{4}\Lambda^0$$

or

$$\boxed{2(n + \Xi^0) = \Sigma^0 + 3\Lambda^0}$$

GELL-MANN OKUBO mass formula.

(9)

N.B. The labels "Ξ^0" etc are convenient ways to refer to a specific member of an SU(3) multiplet, but the relation holds equally between the equivalent members of any other SU(3) multiplet.

The complication in the formula comes about because of the ISOSPIN MIXING in the Σ^0_u state.

To see this, consider the DECUPLET $U = 3/2$ multiplet;

	Δ^-	Σ^-	Ξ^-	Ω^-
U_3	$3/2$	$1/2$	$-1/2$	$-3/2$

Here there is NO MIXING, so the mass shifts are given by

$$\delta m_i = \langle i | H_{MS} | i \rangle = a' + b' U_3$$

i.e. each step in U_3 generates a CONSTANT step in δm, as previously observed.

The formalism derived above allows one to describe BARYON multiplets; for MESONS, we must also consider OCTET- SINGLET mixing.

We suppose that the octet and singlet states are DEGENERATE under H_{VS}, and that H_{MS} lifts the degeneracy.

The states which mix are

- $|\underline{8}, I=0, I_3=0\rangle$
- $|\underline{1}, I=0, I_3=0\rangle$

The ISOVECTOR state $|\underline{8}, I=1, I_3=0\rangle$

(11)

cannot be treated in this way. This is because Y commutes with I_3 and so conserves isospin. Thus matrix elements between states of different isospin VANISH.

This time we use DEGENERATE PERTURBATION THEORY.

In terms of the octet and singlet states, the perturbation matrix is

$$\begin{pmatrix} \langle 8|H_{MS}|8\rangle & \langle 8|H_{MS}|1\rangle \\ \langle 1|H_{MS}|8\rangle & \langle 1|H_{MS}|1\rangle \end{pmatrix}$$

So the secular equation is

$$\begin{vmatrix} M_{88}-m & M_{18} \\ M_{81} & M_{11}-m \end{vmatrix} = 0$$

The matrix is HERMITIAN, so if we choose to make the states REAL, $M_{18} = M_{81}$.

(12)

Solving,

$$(M_{88}-m)(M_{11}-m) - |M_{81}|^2 = 0$$

$$m^2 - (M_{88}+M_{11})m + (M_{88}M_{11} - |M_{81}|^2) = 0$$

giving solutions

$$m_\phi = \tfrac{1}{2}\left\{(M_{88}+M_{11}) + \sqrt{(M_{88}-M_{11})^2 + 4|M_{81}|^2}\right\}$$

$$m_\omega = \tfrac{1}{2}\left\{(M_{88}+M_{11}) - \sqrt{(M_{88}-M_{11})^2 + 4|M_{81}|^2}\right\}$$

where as before m_ϕ and m_ω are labels for the physical states.

However, these equations still involve the variables M_{88}, M_{11} and M_{81}, of which we know how to calculate only M_{88} :-

$$M_{88} = \tfrac{1}{3}(4m_K - m_\rho)$$

(from the GMO formula)

Note also, adding the expressions for m_ϕ and m_ω

(13)

$$m_\phi + m_\omega = M_{88} + M_{11}$$

... but we need a new approach for M_{81}

We introduce the MIXING ANGLE, θ, in terms of which we define two eigenvectors of the perturbation matrix

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

in terms of which

$$|\phi\rangle = \cos\theta\,|8\rangle + \sin\theta\,|1\rangle$$
$$|\omega\rangle = -\sin\theta\,|8\rangle + \cos\theta\,|1\rangle$$

Thus

$$\begin{pmatrix} M_{88} & M_{81} \\ M_{18} & M_{11} \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = m_\phi \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

so

$$M_{88}\cos\theta + M_{81}\sin\theta = m_\phi \cos\theta \qquad [A]$$

and

$$\begin{pmatrix} M_{88} & M_{81} \\ M_{18} & M_{11} \end{pmatrix}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = m_\omega \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

so

$$-M_{88}\sin\theta + M_{81}\cos\theta = -m_\omega\sin\theta \qquad [B]$$

From [A], collecting terms

$$(M_{88} - m_\phi)\cos\theta + M_{81}\sin\theta = 0$$

$$\tan\theta = \frac{m_\phi - M_{88}}{M_{81}} \qquad [A1]$$

From [B]

$$(M_{88} - m_\omega)\sin\theta = M_{81}\cos\theta$$

$$\tan\theta = \frac{M_{81}}{M_{88} - m_\omega} \qquad [B1]$$

Combining [A1] and [B1]

$$\tan^2\theta = \frac{m_\phi - M_{88}}{M_{88} - m_\omega} \qquad [C]$$

and

$$M_{81}^2 = (m_\phi - M_{88})(M_{88} - m_\omega)$$

$$= M_{88}M_{11} - m_\phi m_\omega$$

Notice that in [C] we have replaced one unknown [M_{81}] by another [$\tan\theta$].

UNFORTUNATELY, for mesons these formulae do not work very well. It was quickly found that using m^2 gave better results than using m.

The FEYNMAN PRESCRIPTION for mesons is to use m^2 instead of m. It gains some plausibility from the argument that in boson field theories one deals with m^2, while for fermions one deals with m.

In summary, we have

$$\left.\begin{aligned} M_{88}^2 &= \tfrac{1}{3}\left(4m_K^2 - m_\rho^2\right) \\ m_\phi^2 + m_\omega^2 &= M_{88}^2 + M_{11}^2 \\ \tan^2\theta &= \frac{m_\phi^2 - M_{88}^2}{M_{88}^2 - m_\omega^2} \end{aligned}\right\} \text{MESONS}$$

The labels ϕ, ω, K, ρ refer to positions in the nonets, and could apply to any nonet.

(16)

EXAMPLE: In the 1^- nonet

$m_\rho = 768.3$ MeV $m_\omega = 781.9$ MeV $m_\phi = 1019.4$ MeV

$m_{K^*} = 891.8$ MeV

so

$$M_{88}^2 = 929.33^2 \text{ MeV}^2$$
$$M_{11}^2 = 887.07^2 \text{ MeV}^2$$
$$\theta = 39.83^\circ$$

Octet - singlet mixing alters the flavour composition of the physical states. The flavour wavefunctions for the pure SU(3) states are

$$|8\rangle = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

$$|1\rangle = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

We now determine the "magic" mixing angle θ which gives physical states separated into pure $s\bar{s}$ and non-strange states.

(17)

Choosing θ to make the ϕ state pure $s\bar{s}$

$$s\bar{s} = \cos\theta \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) + \sin\theta \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

so

$$\cos\theta + \sqrt{2}\sin\theta = 0$$

$$\tan\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 35.3°$$

i.e. an angle very close to the phenomenological mixing angle for the ϕ.

MASS SPLITTING IN THE QUARK MODEL

The mass splittings in $SU(3)$ multiplets can also be derived through Quark Model dynamics.

- Assume u and d quarks have the same mass, but that there is an additional increment for s quarks.

- Assume there is a term representing binding potential, spin-spin coupling, etc. We assume this to be FLAVOUR INDEPENDENT, i.e. to have a CONSTANT value for a given multiplet.

Using this scheme, the mass of a meson with quark content $d\bar{s}$ is

$$\langle d\bar{s}|\mathcal{H}|d\bar{s}\rangle = M_0 + d + s$$

while for (e.g.) the pure octet $\underline{I}=0$ state we have

$$\left\langle \frac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s})\middle|\mathcal{H}\middle|\frac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s})\right\rangle = M^0 + \frac{1}{6}(4u+8s)$$

We shall now attempt to retrieve the Gell-Mann Okubo mass formula, previously obtained by symmetry arguments. The "key states" in an SU(3) multiplet are the K^0, π^0, η_8 and η_1. For these we obtain

$$m_{K^0} = \langle d\bar{s}|\mathcal{H}|d\bar{s}\rangle = M_0 + d + s = M_0 + u + s$$

$$m_{\pi^0} = \langle \tfrac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})|\mathcal{H}|\tfrac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})\rangle = M_0 + \tfrac{1}{2}(4u) = M_0 + 2u$$

$$m_{\eta_8} = \langle \tfrac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})|\mathcal{H}|\tfrac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})\rangle = M_0 + \tfrac{1}{6}(4u + 8s)$$

$$m_{\eta_1} = \langle \tfrac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})|\mathcal{H}|\tfrac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})\rangle = M_0 + \tfrac{1}{3}(4u + 2s)$$

We have four equations and three unknowns: M_0, d and s. For the three equations for the octet members, K^0, π^0, and η_8, we can eliminate the unknowns to obtain

$$4\, m_{K^0} - m_{\pi^0} = 3\, m_{\eta_8} .$$

which is the GELL-MANN OKUBO mass formula.

OCTET-SINGLET mixing is dealt with in a very similar way to the simple SU(3) argument.

Labelling the physical states as η and η', we may write

$$\eta = \eta_8 \cos\theta + \eta_1 \sin\theta$$

$$\eta' = -\eta_8 \sin\theta + \eta_1 \cos\theta$$

Inverting,

$$\eta \cos\theta - \eta' \sin\theta = \eta_8$$

Assuming the pure octet mass to be given by the GMO formula, and noting that the physical states η and η' are *orthogonal eigenstates of the mass matrix*

$$\begin{aligned} m_{\eta_8} &= \langle \eta\cos\theta - \eta'\sin\theta | \mathcal{H} | \eta\cos\theta - \eta'\sin\theta \rangle \\ &= \cos^2\theta \langle \eta | \mathcal{H} | \eta \rangle + \sin^2\theta \langle \eta' | \mathcal{H} | \eta' \rangle \\ &= \cos^2\theta \, m_\eta + \sin^2\theta \, m_{\eta'} \end{aligned}$$

Applying the GMO formula and dividing by $\cos^2\theta$

$$(4\, m_{K^0} - m_{\pi^0}) = 3(\cos^2\theta\, m_\eta + \sin^2\theta\, m_{\eta'})$$

$$(4\, m_{K^0} - m_{\pi^0})(1+\tan^2\theta) = 3\, m_\eta + 3\tan^2\theta\, m_{\eta'}$$

Collecting terms

$$(4\, m_{K^0} - m_{\pi^0} - 3\, m_{\eta'})\tan^2\theta = 3\, m_\eta - 4\, m_{K^0} + m_{\pi^0}$$

so

$$\tan^2\theta = \frac{3\, m_\eta - 4\, m_{K^0} + m_{\pi^0}}{4\, m_{K^0} - m_{\pi^0} - 3\, m_{\eta'}} .$$

which is the same result as obtained previously.

Notice that we have obtained the formula in its "m" rather than its "m^2" version.

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CHARM is readily handled in the quark model. For symmetry arguments, it means we should go to SU(4), where our SU(3) relation

$$3 \otimes \bar{3} = 1 \oplus 8$$

becomes

$$4 \otimes \bar{4} = 1 \oplus 15$$

Thus there are 7 new states per spin multiplet:

3 with $c = +1$

		$M(0^-)$ (MeV)		$M(1^-)$ (MeV)
$c\bar{u}$	D^0	1864	D^{*0}	2007
$c\bar{d}$	D^+	1869	D^{*+}	2010
$c\bar{s}$	D_s^+	1968	D_s^{*+}	2110

3 with $c = -1$

$\bar{c}u$	$\bar{D}^0$
$\bar{c}d$	D^-
$\bar{c}s$	D_s^-

and

$c\bar{c}$	η_c	2976	J/ψ	3097

The $c\bar{c}$ state may or may not mix with the non-charmed $q\bar{q}$ states.

The effective c quark mass is CONSIDERABLE.

Although mass splittings are handled in the same way when CHARM is included, the form of the "potential" term must now be more carefully considered. We now have

$$m(q_1, \bar{q}_2) = m_1 + m_2 + \frac{a\,(\underline{\sigma}_1 . \underline{\sigma}_2)}{m_1 m_2}$$

where $\underline{\sigma}_1 . \underline{\sigma}_2$ represents the spin-spin interaction and

$$\underline{\sigma}_1 . \underline{\sigma}_2 = -3 \text{ for } S = 0$$

$$\underline{\sigma}_1 . \underline{\sigma}_2 = +1 \text{ for } S = 1 .$$

A fit to the 0^- and 1^- multiplets yields

$m_u = m_d = 0.31\,\text{GeV}$, $m_s = 0.48\,\text{GeV}$, $m_c = 1.65\,\text{GeV}$, $\alpha = 0.01$

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