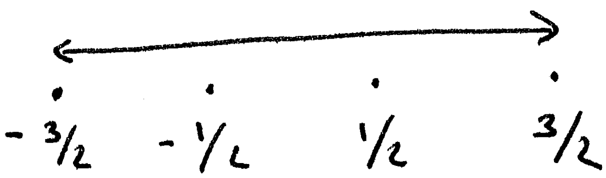
WEIGHT DIAGRAMS

A weight diagram is the SU(3) equivalent of a linear SU(2) multiplet diagram. In SU(2) the structures are simple; for example a ~~3/2~~ $I = 3/2$ multiplet is shown as



For SU(3) more complicated structures are possible, because a given position in the diagram can be occupied by more than one state.

For SU(3) our starting point is that the multiplet structure can be represented by the positions of points $\underline{g}$, where

$$\underline{g} = (I_3, M)$$

are the eigenvalues of the operators

$$\underline{G} = (\underline{I}_3, \underline{G}(\underline{Y}))$$

In more compact notation

$$\underline{G}\,|\alpha, \underline{g}\rangle = \underline{g}\,|\alpha, \underline{g}\rangle$$

where α represents any other quantum numbers.

The weight $\underline{g}$ defines the positions of the allowed states in the (I_3, M) plane.

The MULTIPLICITY of each weight is the number of distinct states associated with that weight.

A multiplet (supermultiplet) is defined by listing all the WEIGHTS and their MULTIPLICITIES for a given multiplet.

We can propose a series of theorems for multiplets.

SHIFT THEOREM

If $|\alpha, \underline{g}\rangle$ is a state of weight $\underline{g}$, then

$I_+ |\alpha, \underline{g}\rangle$

either has weight ZERO, or weight $\underline{g} + \underline{\hat{i}}$ (i.e. it has incremented by 1 in the $\underline{\hat{i}}$ direction.

Similarly

$I_- |\alpha, \underline{g}\rangle$, if non-zero, has weight $\underline{g} - \underline{\hat{i}}$

$U_\pm |\alpha, \underline{g}\rangle$, " " " " " $\underline{g} \pm \underline{\hat{u}}$

$V_\pm |\alpha, \underline{g}\rangle$, " " " " " $\underline{g} \pm \underline{\hat{v}}$

③ Proof.

Consider U_+

We have

$$[\underline{G}, U_\pm] = \pm \underline{\hat{u}}\, U_\pm$$

i.e $\quad \underline{G}\, U_+ = U_+ (\underline{G} + \underline{\hat{u}})$

Applying this to $|\alpha, \underline{g}\rangle$

$$\underline{G}\, U_+ |\alpha, \underline{g}\rangle = U_+ (\underline{G} + \underline{u}) |\alpha, \underline{g}\rangle$$
$$= (\underline{g} + u)\, U_+ |\alpha, \underline{g}\rangle$$

i.e. $U_+ |\alpha, g\rangle$ is a state with weight $\underline{g} + \underline{u}$

REFLECTION THEOREM

The weight diagram of a supermultiplet is SYMMETRIC ~~[illegible]~~ with respect to reflections in the three lines through the origin PERPENDICULAR TO $\underline{\hat{\imath}}$, $\underline{\hat{u}}$ and $\underline{\hat{v}}$

PROOF

A reflection through the line perpendicular to $\underline{\hat{\imath}}$ is equivalent to the rotation

$$P_i = e^{-i\pi I_2}$$

for I_3

Consider how rotations are executed for I_1 and I_3

$$F(\beta) = e^{-i\beta I_2} I_1 e^{i\beta I_2} \qquad (I_1)$$

$$K(\beta) = e^{-i\beta I_2} I_3 e^{i\beta I_2} \qquad (I_3)$$

Differentiating

$$\frac{dF}{d\beta} = e^{-i\beta I_2}(-iI_2)I_1 e^{i\beta I_2} + e^{-i\beta I_2} I_1 e^{i\beta I_2}(iI_2)$$

$$= i e^{-i\beta I_2}[I_1, I_2] e^{i\beta I_2}$$

$$= -e^{-i\beta I_2} I_3 e^{i\beta I_2} \quad = -K(\beta)$$

$$\cancel{\frac{dK}{d\beta} = e^{-i\beta I_2}(-iI_2)I_3}$$

$$\frac{dK}{d\beta} = e^{-i\beta I_2}(-iI_2)I_3 e^{i\beta I_2} + e^{-i\beta I_2} I_3 e^{i\beta I_2}(iI_2)$$

$$= i e^{-i\beta I_2}[I_3, I_2] e^{i\beta I_2}$$

$$= e^{-i\beta I_2} I_1 e^{i\beta I_2} = F(\beta)$$

so

$$\boxed{\begin{aligned} \frac{dF}{d\beta} &= -K(\beta) \\ \frac{dK}{d\beta} &= F(\beta) \end{aligned}}$$

Now

$$F(\beta = 0) = I_1$$

$$K(\beta = 0) = I_3$$

⑤ so

$$F(\beta) = I_1 \cos\beta - I_3 \sin\beta$$
$$K(\beta) = I_1 \sin\beta + I_3 \cos\beta$$

$\therefore$ For $\beta = \pi$

$$e^{-i\pi I_2} I_1 e^{i\pi I_2} = F(\pi) = -I_1$$
$$\boxed{e^{-i\pi I_2} I_3 e^{i\pi I_2} = K(\pi) = -I_3}$$

Now, if

$$P_i^{-1} I_3 P_i = -I_3$$

and $$P_i^{-1} M P_i = M \quad (\text{since } I_2 \text{ commutes with } M)$$

then if $|\alpha, \underline{g}\rangle$ is in the supermultiplet

$P_i |\alpha, \underline{g}\rangle$ is also in the supermultiplet

and has eigenvalues

$$(-i_3, m)$$

Note that $P_i |\alpha, \underline{g}\rangle$ cannot be zero, since the operator P_i is UNITARY.

Similar proofs follow for $\underline{U}$ and $\underline{V}$.

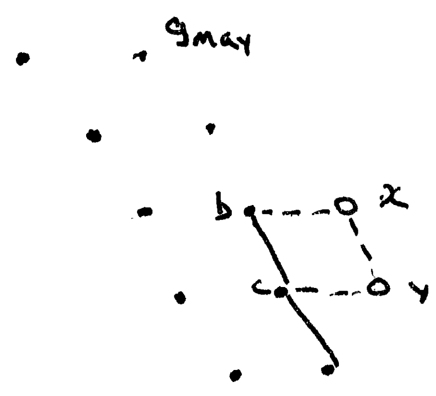
HIGHEST WEIGHT THEOREM

We define the state with the HIGHEST WEIGHT in a supermultiplet as the state with highest value of M. If there is more than 1, then highest value of I_3. This state has UNIT multiplicity

(Proof is lengthy.)

As a result, the extremes of the supermultiplet form pairs symmetric about the $\underline{\hat{I}}$, $\underline{\hat{u}}$ and $\underline{\hat{v}}$ axes.

EDGES OF SUPERMULTIPLET FORM STRAIGHT LINES LINKING STATES WITH UNIT MULTIPLICITY



Suppose state b is non-empty, and has multiplicity 1.

Additionally, suppose state x is empty.

(7) We may reach $\underline{c}$ from $\underline{b}$ as follows:

$$|c\rangle_1 = U_-|b\rangle$$
$$|c\rangle_2 = I_+V_+|b\rangle$$
$$|c\rangle_3 = V_+I_+|b\rangle$$

Note that $[V_+, I_+] = U_-$

so these are not independent

$$V_+I_+|b\rangle - I_+V_+|b\rangle = U_-|b\rangle$$

and, by hypothesis $|x\rangle = 0$, so

$$I_+|b\rangle = 0$$

and

$$|c\rangle = |c\rangle_2 = |c\rangle_1$$
$$|c\rangle_3 = 0$$

Therefore $|c\rangle$ has multiplicity 1.

What about $|y\rangle$?

Suppose it is non-empty. Then

$$I_-|y\rangle = |c\rangle, \quad \langle y|I_-^\dagger = \langle c|$$

So, as $U_-|b\rangle = |c\rangle$

$$\langle c|c\rangle = \langle y|I_-^\dagger U_-|b\rangle$$
$$= \langle y|I_+U_-|b\rangle$$
$$= \langle y|U_-I_+|b\rangle$$
$$= 0$$

But $|c\rangle \neq 0$

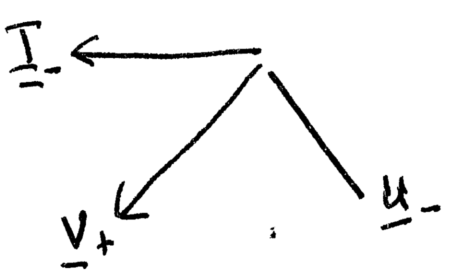
so $|y\rangle$ must be equal to zero.

We can therefore start with state $|g_{max}\rangle$ and sweep down side by successive application of this principle.

MULTIPLICITY OF INNER LAYERS

~~B~~. Similar arguments can be used to determine the multiplicities of the inner layers.

Starting from the state g_{max}, we may consider how to reach the state $\underline{g} + \underline{v}$



Firstly, not all the states

$I_- |\alpha, g_{max}\rangle$

$V_+ |\alpha, g_{max}\rangle$

$U_- |\alpha, g_{max}\rangle$

can be zero

We may reach $\underline{g} + \underline{v}$ as follows:

$V_+ |\alpha, g_{max}\rangle$, $I_- U_- |\alpha, g_{max}\rangle$, $U_- I_- |\alpha, g_{max}\rangle$

(9) However, these are not independent, as

$$I_- U_- |g_{max}\rangle - U_- I |g_{max}\rangle = -V_+ |g_{max}\rangle$$

So there are at most TWO independent paths. If either $I_- |g_{max}\rangle$ or $U_- |g_{max}\rangle$ is equal to ZERO, there is only one.

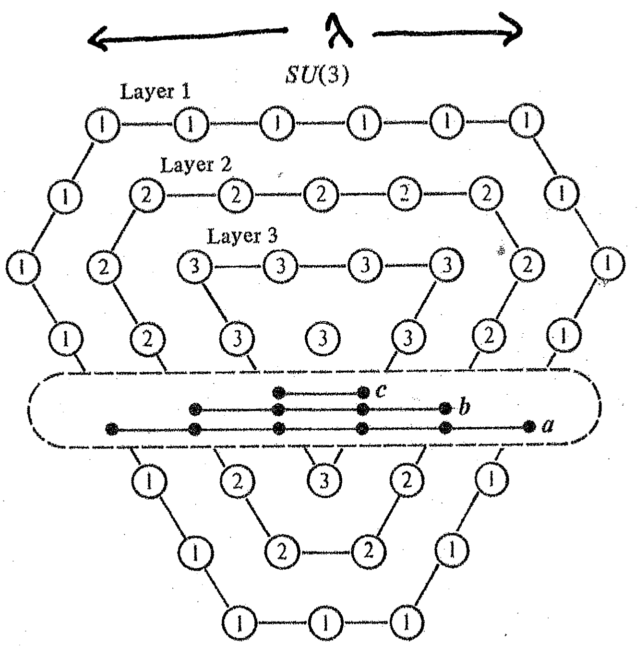


Fig. 10.7. The supermultiplet 81 with the layers indicated. Within the dashed line all of the states with a particular M are displayed and grouped into isospin multiplets with $I = \frac{1}{2}, \frac{3}{2}$ and $\frac{5}{2}$.

$\longleftarrow \mu \longrightarrow$

By repeated application of these principles we find successive layers of the supermultiplet have larger multiplicities until a TRIANGULAR layer is reached, ~~when~~ after which successive layers have the SAME weights.

Note also that only 2 dimensions are required to characterize the multiplet; the largest (λ) and smallest (μ) values of $(2I_3+1)$.