Worksheet 1 — Mathcad

Week 1, Session 1

You have already used Mathcad much earlier in your course and so this first worksheet has a dual purpose. It is intended to remind you of some introductory topics as well as introducing you to some new material.

1. The "Equals" Sign in Mathcad

In Mathcad there are a number of "equals" signs which have different meanings. It is important to understand the differences. The three most important ones are:

$=$ which means evaluate and display the expression

$:=$ which means assign 'locally'

$\equiv$ which means assign 'globally'.

The local and global assignment operations, $:=$ and $\equiv$ respectively, are used for assigning values and functions to variables. Global variables can be defined and used at any point in a worksheet whereas local variables can only be used after they have been defined (to the right on the same line or below the definition). The value of a local variable can be changed at various places in a worksheet.

Evaluate the total energy of an object with rest mass m and velocity v using the following formula:

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

where $m = 1$, $v = 1 \times 10^8$ and $c = 3 \times 10^8$. Use the local assignment operator $:=$ to define m, v, c and E. Try moving the definitions of m, v and c below that of E. Now try changing the local assignments of the variables to global assignments.

2. Defining Functions

Assignments can also be used to define functions of one or more variables in Mathcad. Try defining the error function:

$$\mathrm{err}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Notice that neither x, nor t have to be defined, they are dummy variables. The meaning of this definition is exactly the same as in a formal mathematical document. Evaluate err(1), err(2) and err(3), and check that they are reasonable.

3. Plotting Functions

Define the function

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}.$$

Plot the function over the range $-20 < x < 20$ in steps of 0.5. Notice the discontinuity at $x = 0$. This is because Mathcad, or any similar package, incorrectly evaluates $\sin(0)/0$ as 0 instead of the limit value 1. To overcome this problem, change your definition of sinc to use Mathcad's *if* construction so that it looks like:

$$\operatorname{sinc}(x) := \text{if}\left(x = 0, 1, \frac{\sin(x)}{x}\right)$$

The '=' inside the *if* function is Mathcad's logical equals operator and is displayed in bold on your screen. The *if* function is useful for making more complex functions from simpler functions.

From now on it is probably worth creating a separate Mathcad worksheet for each independent exercise, and remember to save your worksheets regularly to the floppy disk.

4. Vectors and Plotting Arrays

Create two 1–dimensional arrays, one containing values in the range -20 to 20 in steps of 0.5 and the other containing values of the sinc function at these points.

Arrays can be plotted in a similar way to functions. Plot the array of $\operatorname{sinc}(x)$ values against the array of x values you have created. Experiment with some of the plotting options. This technique can be used to create all sorts of graphs useful for analysis and presentation of results.

The advantage of plotting with arrays rather than functions is that Mathcad can generate 2–dimensional plots from 2–dimensional arrays. Start by generating a 2–dimensional array containing values of the function:

$$F(x, y) = \operatorname{sinc}\left(\sqrt{x^2 + y^2}\right)$$

over the range $-20 \leq x, y \leq 20$. Then ask for a surface plot from the menu, or graphics palette, and use this array as the input to the plot. You can experiment with various options to study the graph in many ways.

An alternative to surface plots are contour plots, which make plotting some functions clearer. Try generating a contour plot with the same function, and again experiment with the various display options. In fact, this is not a very good function to display using a contour plot, so you might like to think of a different function which would give a more interesting contour plot.

5. Equation Evaluation

Here is an example which uses some of Mathcad's ability to solve equations numerically.

The Breit-Wigner distribution was originally introduced to describe the cross section of resonant nuclear scattering which had been derived from the transition probability of a resonant state with known lifetime. In particle physics the Breit-Wigner function is used to describe the decay of a resonance to other particles in terms of the mass and width of the resonance. The Breit-Wigner amplitude is written as

$$BW(m) = \frac{m_0\Gamma}{m_0^2 - m^2 - im_0\Gamma}$$

which is a complex number. m_0 is the pole mass of the resonance, m is the mass of the system to which the particle has decayed to and Γ is the decay width of the resonance. The observed resonance has a shape given by

$$Int(m) = |BW(m)|^2$$

In addition to studying the Intensity of the resonance its phase variation ϕ defined as

$$\phi(m) = tan^{-1}(\frac{Im(BW(m))}{Re(BW(m)}$$

can also be studied. The definition of a resonance is that the phase of the Breit Wigner amplitude should go through 90 degrees at the pole mass m_0.

Define the Breit Wigner amplitude and plot the observed resonance shape (Int) for a resonance with $m_0 = 1000$ MeV and a width $\Gamma = 20$ MeV for a mass range m from 950 to 1050 MeV. *Comment on the observed distribution.* Determine the full width half maximum (FWHM) of the distribution either by trial and error or by using the Mathcad function Minerr. *Comment on how the FWHM compares to* Γ

Determine the phase variation ϕ of the resonance as a function of mass (you may find the Mathcad functions atan2, Re and Im useful). *Comment on the phase variation observed.*

A more familiar distribution is the Normal or Gaussian Distribution defined as

$$Int(m) = exp^{\frac{-(m-m_0)^2}{2\sigma^2}}$$

Plot the Gaussian distribution using the same parameters as previously and using $\sigma = \Gamma$. *Determine the FWHM of the Gaussian distribution and compare it with what you expect from a Gaussian distribution.*

Change the value of σ such that the Gaussian distribution has the same FWHM as the Breit-Wigner distribution i.e. determine the value of *scale* in $\sigma = \Gamma/scale$ such that the FWHM is the same for Gaussian distribution and the Breit-Wigner Distribution. *comment on the value of scale.* Plot the Breit Wigner and the Gaussian distribution on the same graph. *Comment on the difference between the two distributions.*

Week 1, Session 2

6. An introduction to non-linear systems

We have chosen an example which illustrates some of the properties of non-linear systems, such as bifurcations and chaos. This is not something that would be easy to do with pen and paper, as it requires a large amount of calculation. You may have seen this problem before, this is probably because it nicely illustrates the power of computers in solving numerical problems.

Consider the recurrence relation:

$$x_{n+1} = kx_n(1 - x_n)$$

where $0 < x_n < 1$ and k lies in the range 0 to 4. This could be used as a greatly simplified model of, for example, animal population dynamics, *e.g.* if N_0 is the number of green-fly needed to eat all the leaves in a rose garden one summer, and suitable assumptions are made about the way rose trees recover, then $N_0 x_n$ could represent the number of green-fly in the n^{th} summer.

Write a Mathcad worksheet to investigate the short-term and long-term behaviour of x_n (say with $n \sim 1000$) with a fixed value of k and an arbitrary starting value x_0. Observe the behaviour as the starting value is changed. Then try varying the value of k over the full range, and again investigate the long-term behaviour of x_n. Values of k around and above 3 are the most interesting. Spend a little time investigating the behaviour until you understand most of what is going on.

You will find that for some values of k, x_n tends to a unique value $X(k)$, while for others it 'cycles' around a set of values $X_i(k)$, where i runs over a range 1 to N and N is a function of k. You should notice distinct changes in the behaviour of x_n when k passes through the following values:

1.000, 3.000, 3.4495, 3.5441 ...., 3.5699.

Make a plot of the values $X(k)$, or $X_i(k)$, as a function of k. It is probably not immediately obvious how to do this, but with your knowledge of Mathcad and a little thought, it should be possible. It may be useful to make separate plots showing details around regions of particular interest. You should use these plots to confirm the following points:

- A bifurcation occurs at all of the points in the list above (except 1.000). A bifurcation means a point where the system converts from having one stable solution to having more than one. In general the original single stable solution becomes unstable.
- The number of stable solutions doubles at each bifurcation point.
- The 'analytical' solution, $x = (k-1)/k$, is valid for all $k > 1$, and stable for $k < 3.000$ but unstable for larger values of k.
- In the region $3.5699 < k < 4.0$ there is no discernible long-term pattern to x_n. This region is an example of chaotic behaviour.
- In the chaotic region, the sequences x_n generated by two very nearly equal values of x_0 bear no resemblance to each other after just a few iterations.
- There is structure within the 'chaos', *i.e.* there are regions which are not highly populated within the x_n versus k space.

This exercise illustrates the general properties of a non-linear system, and shows how a simple equation can lead to a rich variety of behaviour.

7. Further Work(Optional)

If you have time, experiment further with other features of Mathcad, and try to think of some interesting problems from your undergraduate courses that you could solve or illustrate with Mathcad.

Make sure you are familiar with all the techniques used this week, as many of them will be used regularly in the rest of the classes.