

Worksheet 2 — Classical Vibrations and Waves

Introduction

You will have encountered many different examples of wave phenomena in your courses and should be familiar with most of the mathematics illustrated in this set of exercises. Some of the problems covered can be easily solved analytically and, if you feel inclined, you can derive the analytical solutions to see how they compare. However, the point of using Mathcad here is that it allows an easy way to visualise wave phenomena, and it is also simple to change the input parameters in order to see how this changes the result, and therefore it should help to give more of a feel for what is going on. Also, in cases where a large number of calculations are needed to plot the behaviour of a system, it is clearly best to use a computer, and Mathcad provides a simple interface for this heavy computing.

You should work through the problems, eventually handing in your Mathcad worksheet for each problem with the parameters set at interesting cases. You need not display a graph for *every* set of parameters you tried but enough to illustrate the effects you wish to discuss. Also, you should note down the behaviour observed when changing the input parameters to each problem, with explanation of the phenomena. Do this at appropriate places within the worksheet and be sure to summarise your findings at the end of each section. Particular points and questions that you should answer to obtain marks are *picked out in italics*.

Week 2, Session 1

1 The Damped Harmonic Oscillator

The damped harmonic oscillator is probably the most frequently encountered type of vibration in classical physics. Almost any small vibration can be approximated as being of this type. You have probably worked through many pendulum or ‘mass on a spring’ problems which either involve simple harmonic motion (when there is no damping) or damped harmonic motion. It also occurs in basic resonant electrical circuits.

A convenient form of the equation of motion for a damped harmonic oscillator is:

$$\frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + \omega^2 x = 0 \quad (1)$$

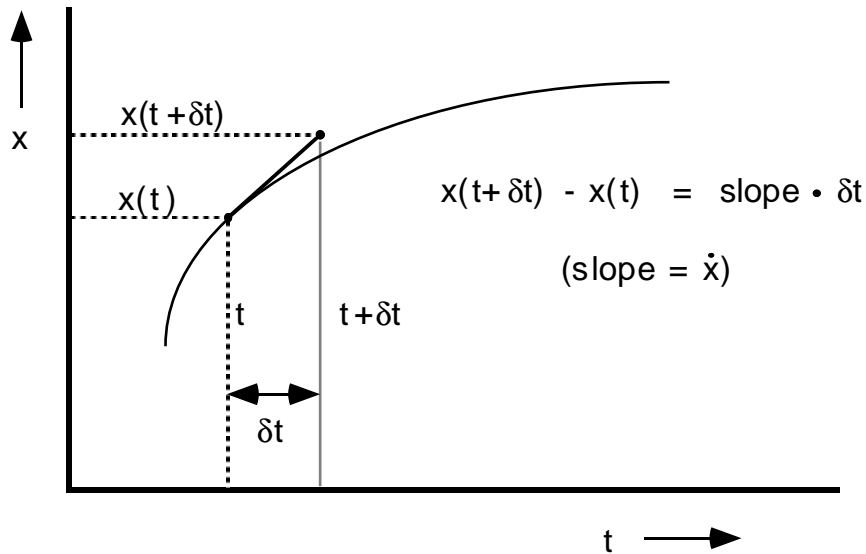


Figure 1: First order approximation to solve a differential equation.

where ω is the natural angular frequency of the oscillator, and λ is the damping factor — if $\lambda = 0$ the equation reduces to the simple harmonic oscillator.

Although we could solve this particular problem analytically, there are many differential equation problems where an analytical solution is impossible, so in those cases we have to use a numerical approximation method. To illustrate these methods we will solve the oscillator problem in this way. The method used below is the simplest and most crude numerical solution.

To solve a differential equation problem of this sort, we need to specify two initial conditions such as: At time $t = 0$, then $x = 1$ and $dx/dt = 0$, or:

$$x(0) = 1, \quad \dot{x}(0) = 0$$

Knowing (or setting) these initial conditions, we can *approximate* the values at a time δt later as:

$$x(\delta t) = x(0) + \delta t \cdot \dot{x}(0), \quad \dot{x}(\delta t) = \dot{x}(0) + \delta t \cdot \ddot{x}(0)$$

or, more generally:

$$x(t + \delta t) = x(t) + \delta t \cdot \dot{x}(t), \quad \dot{x}(t + \delta t) = \dot{x}(t) + \delta t \cdot \ddot{x}(t)$$

This is the simplest (first order) approximation that can be made to derive numerically the shape of a curve. The technique is illustrated in Figure 1.

Now, using equation 1, we can change these equations to:

$$x(t + \delta t) = x(t) + \delta t \cdot \dot{x}(t) \tag{2}$$

$$\dot{x}(t + \delta t) = \dot{x}(t) - \delta t(\lambda \dot{x}(t) + \omega^2 x(t)) \tag{3}$$

With these equations, and knowing the initial conditions, we can iteratively step through to the position at any time. Obviously, the smaller the value of δt , the better the approximation will be. You can also see how this technique could be easily extended to a differential equation problem of higher order.

Mathcad can perform this sort of iteration, where subsequent values of a variable depend solely on the previous values. To do this problem you will need to construct a vector of seeded iterations. Use variables for ω and λ so that you can easily change their values later. Start with $\omega = 0.1$ and $\lambda = 0.03$. You will also need to choose the initial values for x and \dot{x} — try 1 and 0 to start with.

Calculate the values of x over the time range $t = 0$ –400, using time steps of 1, and plot them on a graph. This should give you the classic motion of a damped oscillator, which consists of a decaying sine wave in an exponential envelope. Once you have achieved this, you can then investigate the behaviour with different initial conditions, and different frequencies and damping factors.

Try varying ω in the range 0.03 to 0.1 with $\lambda = 0.03$, and *make brief observations on the shape of the oscillation as the frequency changes*. Now try varying λ in the range 0.03 to 0.3 with $\omega = 0.1$. *Comment on the shape of the curve, and in particular investigate the behaviour of the system as λ approaches 0.2. What phenomenon is occurring here?* The minimum value of the curve is a useful quantity to observe in this study.

Finally plot what happens when you try $\omega = 0.2$, $\lambda = 0.03$ and $\omega = 0.1$, $\lambda = 0.01$. This behaviour is clearly wrong, as the oscillation should not increase. This is an artifact of the approximation method, and the approximation is clearly not good for these values of the input parameters. As stated above, this is the most crude approximation technique for differential equations. It could be improved by reducing the step size between iterations, and you might like to try this to see what effect it has. To really improve matters, you have to use a better approximation method using higher order differentials. An example of this is the Runge-Kutta method, which is a fourth order calculation. This technique is often built into mathematical packages so that it can be easily applied to any differential equation problem.

2 Forced Oscillations and Resonance

This is a variation on the previous problem. In this case we have the same sort of oscillator, only now it is subjected to a small harmonic excitation at about the same frequency as the natural frequency of the system. This is also a very common situation, an everyday example of which is the tuning circuit of a radio, where a resonant circuit is excited by electro-magnetic waves incident on the aerial.

The general equation for forced oscillations is:

$$\frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + \omega^2 x = F \sin(\rho\omega t + \delta) \quad (4)$$

You should be able to work out how to introduce this extra term into the numerical approximations of equations 2 and 3. F represents the amplitude of the excitation. A good value for this is 0.01, so use this throughout the problem. We have used the ρ term to conveniently express the frequency of the excitation as a ratio to that of the oscillator, ω . Put this in as a variable as we will want to see how the system responds as the frequency of the input changes. The phase of the input δ can be set at 0.

You can easily duplicate and adapt the previous Mathcad document for this problem, adding the extra term into the iteration, and this time it is useful to let time run from 0 to 1000, in the same steps, and using the original input values of $\omega = 0.1$ and $\lambda = 0.03$.

Start with $\rho = 1.0$, *i.e.* the frequency of the input is the same as that of the oscillator. You should see that the system quickly settles into a sine wave of constant amplitude. Note in particular that this amplitude is a lot larger than the amplitude of the input excitation.

Now try varying ρ in the range 0.5 to 1.5 and note down your observations of the system's behaviour at different excitation frequencies. Note that the start of the motion can be quite messy. This illustrates a phenomenon known as transient behaviour. When the excitation starts the system is not near an equilibrium state, and this produces non-resonant waveforms, which are eventually damped out, hence the name transient.

It is useful to investigate how the resonant behaviour changes with different damping factors. Firstly you must be able to measure the amplitude of the system's response under different conditions. This is most easily achieved by looking at the maximum value of the curve as the system approaches a steady sine wave. Find the magnitude of the response as you vary ρ in the range 0.5–1.5, in steps of 0.1 to start with. Try this with λ values of 0.015, 0.03 and 0.06. You might consider extending the time interval for $\lambda = 0.015$, since the transients take longer to die out in a lightly damped system. Finally plot these resonant responses as a function of ρ separately for each value of λ . If your document doesn't take too long to calculate, you should try using a finer increment for ρ . *Comment on the shape and form of the resonant behaviour seen, and how it compares between the differently damped systems.*

3 Illustration of the Development of Wave Packets

When dealing with waves, we often simplify matters by looking at one continuous running wave of a definite frequency. However, it is not possible to describe a single burst of sound or light using a single running wave. To produce this sort of effect, we have to combine several waves of different frequencies and phases which together make up what is known as a wave-packet. In fact the same applies to the motion of individual particles, which in wave mechanics must be described as the superposition of particle waves of different frequencies.

To try to see what a wave-packet looks like, and see how it develops in time, we can generate a simple example using the superposition of waves given by this formula:

$$W(x, t) = \sum_n A_n \cos(k_n x - \omega_n t) \quad (5)$$

Note that ω is in general a function of k — we will later be defining and changing this relationship, so in your document you should form this relationship as a separate equation.

We shall look at a wave-packet made up from components with wave-numbers

$$k_n = 5.0, 5.1, 5.2 \dots 7.0.$$

We also need to choose the components A_n for each wave-number. Set up two versions, one with constant components $A_n = 1$ and one with Gaussian components $G_n = \exp(-5(k_n - 6)^2)$. These components are illustrated in Figure 2. Note that these Gaussian components are centred on the middle wave-number 6.0, and reduce to close to zero at the limit of wave-numbers used.

Before seeing what these wave-packets look like, we must choose the relationship between ω and k . This is known as the dispersion relationship. For electro-magnetic waves, then the relationship is:

$$\omega(k) = k \times v \quad (6)$$

where v is the velocity (the speed of light in this case). For the purposes of our example, it is convenient to set this velocity to unity.

So now you should be able to plot the shape of the wave-packet $W(x, t)$ on a 1-dimensional graph in distance x . Do this at time $t = 0$ for values of x in the range $-10 < x < 20$, choosing suitable intervals to see the pattern clearly. Do this first of all with the constant component values A_n . You should see a fundamental wave

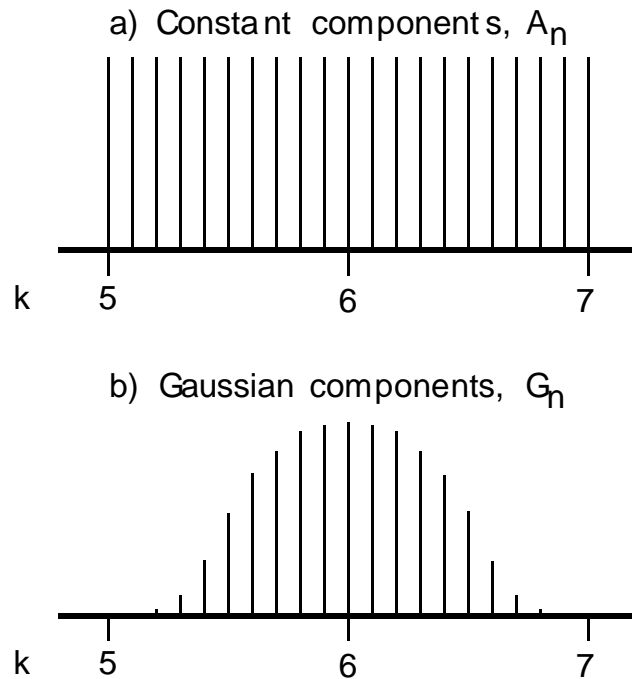


Figure 2: Wavenumber components used to build up the wave-packets.

of wavelength about 1 unit which is contained in an envelope which is large around $x = 0$. *Try to justify the fundamental wavelength of the pattern seen.*

This plot illustrates a wave-packet where there is a burst of large activity around one point. However, you should be able to see from the tails of the wave-packet that there is still much messy activity before and after the burst. This is not surprising as we have only chosen some arbitrary components, which do not produce a neat wave-packet. Now try using the Gaussian components. This time you should see a cleaner Gaussian burst of activity which has little disturbance to either side.

Using the Gaussian wave-packet, try animating the development of the wave as time runs from 0 to 15 (using the FRAME variable provided by Mathcad). You should see that the shape of the disturbance does not change at all, but just shifts along with time. *Measure and justify the speed at which the wave-packet moves.* Now go back to the wave-packet generated by constant components and see how it develops. *Why does this pattern not change with time?*

Now go back to $t = 0$ but display the x range -10 to 100. You should see that the wave-packet appears to be repeated. This is due to the fact that we are only using a finite number of components superimposed to make a wave packet — in a real wave-packet there is a continuum of frequencies superimposed to make up a single burst of activity. *Taking this idea as a basis, explain how you would create components that would make the repetition less frequent.*

Think about how you might be able to apply Fourier analysis to the above observations, and try to explain:

1. The shape of the wave-packet with Gaussian components.
2. The shape of the wave-packet with constant components.
3. The repetition of packets.

in terms of Fourier Transforms and Convolution Theory.

We can use this repeating feature to see how the wave develops after many time cycles. Choose a large value for t , say about 10000, and try to find one of the wave-packets. Note that the wave-packet still has exactly the same shape after travelling for a large distance. The fact that no deformation is seen is a feature of non-dispersive waves, and only occurs when the dispersion relation is of the form of equation 6 above. Next we shall see what happens with a different dispersion relation.

Waves on a pond are a well known phenomena where the dispersion relation takes a different form. In this case, it is:

$$\omega(k) = v \times k^{0.5}$$

Setting the velocity v to unity again for ease, try putting this dispersion relation into your wave simulation, using the Gaussian wave-packet. At $t = 0$ the wave should look exactly the same as before, but now see what happens if you animate it over the time 0 to 50. Firstly it should be obvious that the wave-packet is no longer travelling at the same speed as before. *Try to estimate the speed again.* Also you should be able to see that the internal wave phase is *progressing* through the wave-packet. That is, the internal waves are still travelling at the speed of the individual travelling waves, but the wave-packet is moving more slowly.

The theory behind this tells us that the internal waves should travel at what is known as the phase velocity, whereas the wave-packet travels at the group velocity. To find the phase and group velocity for any wave you need the dispersion relation $\omega(k)$, then:

$$\text{Phase Velocity} = \frac{\omega(k)}{k} \tag{7}$$

$$\text{Group Velocity} = \frac{d\omega(k)}{dk} \tag{8}$$

Check these equations with your observations for both the original dispersion relation, and the water waves case.

Finally try putting the time up to 1000 and then 10000 with the water waves, and see how the wave-packet develops over a long period. Note down your observations. You should by now have understood what makes dispersive and non-dispersive waves different.

Week 3, Session 1

4 Vibrations on a Guitar String by Fourier Series

Here is another case where the basic physics is simple, but having a computer to do the number-crunching is essential. The normal modes of a vibrating string should be very familiar to you, and a guitar string is no different to any other string in this sense. However, as in any musical instrument, it is the proportion of each of the harmonics, and their lifetime that determines the quality of sound. One of the major factors in determining which harmonics are present is the way in which an instrument is played. In the case of a guitar, the distance of the point where it is plucked from the end of the string is critical.

For a string of length L vibrating in the n^{th} normal mode, the displacement, y , of the string at a point x from one end at any time t (assuming no damping) is given by:

$$y(x, t) = C \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right) \quad (9)$$

We have assumed here a dispersion relation of the form $\omega = vk$. Make sure you understand this formula before you continue, and investigate it with Mathcad if you feel uncertain of what it means, then *comment on the meaning of each of the terms in the equation*.

When the string is plucked, the initial displacement can be resolved into Fourier series components in order to pick out which harmonics are being excited. However, we should also introduce a damping term — again from section 1 you can guess that this damping will be exponential. This means that for each component, the equation becomes:

$$y(x, t) = C \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right) \exp(-Dt) \quad (10)$$

So now all that needs to be done is to determine the Fourier components, C , from the initial position, and sum the component waves. If the position of the string at $t = 0$ is given by $g(x)$, then the Fourier components are given by:

$$C = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

With the equations above you should now be able to define a function $F(x, t)$ which gives the position of the string at any time and distance from one end. You will need to choose some parameters, and these are a reasonable set:

- The velocity $v = 1$ for ease.
- The number of Fourier components to sum: $n = 1, 2..10$ is reasonable to start with.
- The input function $g(x)$ due to plucking the string. Try something like a couple of straight lines from $x = 0, F = 0$ and $x = L, F = 0$ to a point $x = L/10, F = 1$. This is illustrated in Figure 3.
- The damping terms. Make these vary between normal modes — in general for instruments, the higher frequency modes will be damped out more quickly. Our example will be exaggerated in order to save time. Try:

$$D_n = \frac{n^2}{1000}$$

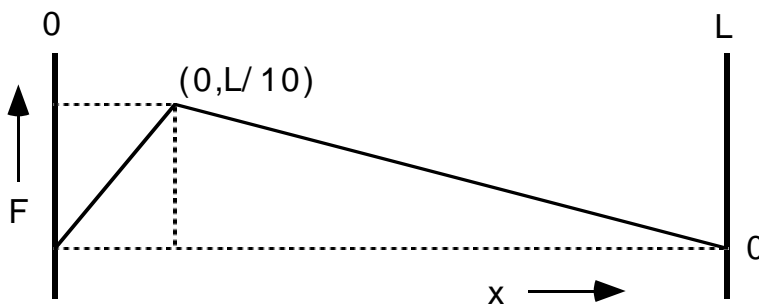


Figure 3: Example of a possible ‘plucking’ function.

Now you have the function $F(x, t)$, plot the position of the string at various times. Again using the animation facility of Mathcad, you can look at the vibration both in the short term (over a couple of fundamental cycles) and after many cycles. *Spend some time exploring the behaviour of the string, varying some of the input parameters, and then note down your observations. In particular contrast the shape of the string at early times and late times and explain the difference.*