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Fermat's principle of least time in the presence of uniformly moving boundaries and media

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The refraction of a light ray by a homogeneous, isotropic and non-dispersive transparent material half-space in uniform rectilinear motion is investigated theoretically. The approach is an amalgamation of the original Fermat's principle and the fact that an isotropic optical medium at rest becomes optically anisotropic in a frame where the medium is moving at a constant velocity. Two cases of motion are considered: a) the material half-space is moving parallel to the interface; b) the material half-space is moving perpendicular to the interface. In each case, a detailed analysis of the obtained refraction formula is provided, and in the latter case, an intriguing backward refraction of light is noticed and thoroughly discussed. The results confirm the validity of Fermat's principle when the optical media and the boundaries between them are moving at relativistic speeds.

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I. INTRODUCTION

The variational approach in geometrical optics was established in 1657 when the famous French mathematician Pierre de Fermat provided a mathematical proof that the straight line is not the fastest way for light to traverse between two optical media [1, 2, 3]. This simple and new way of looking at the propagation of light was already empirically noticed by Hero of Alexandria fifteen centuries before Fermat by observing that the path of the light between two points upon reflection from a plane mirror is the shortest possible path. Fermat's insightful and intriguing idea that "the Nature always acts in the shortest ways" has influenced generations of scientists, like Maupertius, Euler, Lagrange, Hamilton and Feynman, to successively translate classical and quantum physics into the language of variational calculus [1, 4, 5, 6]. Today, the practical value of Fermat's principle is immense, ranging from applications in geophysics, oceanography and material physics on one side, to general relativity and astrophysics on the other [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

The modern formulation of Fermat's principle states that a light in going between two fixed points traverses an optical path length that is stationary with respect to the variations of that path. In most cases, this formulation reduces to the original one that the actual path between the two points is the path which requires the shortest time [21]. An accurate description of the process beyond Fermat's principle is provided by the modern quantum electrodynamics, where the light is considered to be made out of photons. Accordingly, the photon takes all the available paths in going between two fixed points, each path at a certain probability, and the principle of least time follows as an approximation of this argument [22, 23, 24].

Fermat's principle has been initially established for optical systems whose properties are time-independent, and, as such, has entered the mainstream of the standard undergraduate physics curriculum through the derivations of the well-known laws of reflection and refraction of light at a smooth stationary interface between two optical materials at rest [22]. Recently, Fermat's principle was proven valid for optical systems with time-dependent parameters, incorporating the situations of light propagation in uniformly moving optical media [25, 26, 27]. It was also shown that Einstein's law of reflection of light from a plane mirror in uniform rectilinear motion follows from Fermat's principle and the constant speed of light postulate, providing an indirect proof of the validity of Fermat's principle when the boundaries upon which the light is reflected are moving at relativistic speeds [28]. Apart from the latter example, it seems that the problems of light propagation involving uniformly moving boundaries and media have never been attacked directly by using Fermat's principle of least time.

Motivated by the latest extensions to non-stationary cases, in this paper we will use Fermat's principle to derive the formulas for the law of refraction of a light ray in two specific situations when the boundaries and the media are moving at constant velocities. We will limit our analysis to optical media that are homogeneous, isotropic and nonconducting in their rest frames of reference. Also, we consider the media to be non-dispersive at rest, in the sense that the correcting terms due to dispersion are so small that can be neglected.

We begin our discussion in Sec. II by showing that an optical medium, isotropic in its rest frame, exhibits an optical anisotropy with respect to the frame where the medium is in uniform rectilinear motion. We derive the expression for the speed of the photon in the moving medium and show that it depends on the angle between the light ray and the velocity of the medium. We de-

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scribe this relativistically induced optical anisotropy in a more convenient way, by introducing an effective refractive index of the moving medium, defined as a ratio between the speed of the photon in vacuum and the speed of the photon in the moving medium. The notion that the photon will "see" the moving medium as a stationary medium having an effective refractive index different from its refractive index at rest, is incorporated into Fermat's principle in Sec. III and IV to investigate the refraction of the photon incident from a vacuum half-space upon a uniformly moving material half-space. In Sec. III we investigate the situation when the material half-space is moving parallel to the interface, and in Sec. IV we take the material half-space to move perpendicularly to the interface. In both cases, we consider the plane of incidence to be normal to the vacuum-material interface and parallel to the velocity of the medium.

A summary of the results and a comparison with other similar treatments are given in Sec. V, where the possibility to extend the method to more complicated configurations is also suggested. Alternative derivation of the refraction law when the material half-space is moving perpendicular to the interface is presented in Appendix A. Finally, in Appendix B, the conditions for the appearance of the backward refraction are derived analytically.

II. AN OPTICAL ANISOTROPY OF A TRANSPARENT MEDIUM INDUCED BY ITS UNIFORM MOTION

If a flash of light is emitted from a fixed point source in an empty space (a vacuum), the outgoing photons will move radially on a spherical wavefront expanding at a speed of light c [29]. With respect to an observer from a frame that moves in a straight line at a constant velocity, the wavefront of the pulse will remain sphere, expanding at the same constant speed c in all directions [30]. The result is a corollary of the second postulate of special relativity that the speed of light in a vacuum is the same in all inertial frames of reference regardless of the motion of the light source [31].

The situation is changed if the expansion of the pulse is taking place in a material medium [32, 33]. If we consider a homogeneous, isotropic and non-dispersive transparent medium at rest with respect to S'-frame, and if we treat the problem two-dimensionally, the equation that describes the evolution of the pulse is:

$$x'^2 + y'^2 = (c/n)^2 t'^2, \tag{1}$$

where n is the refractive index of the material at rest. This is an equation of an expanding circle whose radius at time t' is $(c/n)t'$. By using the Lorentz transformation:

$$x' = \gamma(x - ut),\ y' = y,\ t' = \gamma(t - ux/c^2), \tag{2}$$

with $\gamma = (1 - u^2/c^2)^{-1/2}$, we can make a transition to S-frame where the medium is moving at a constant speed

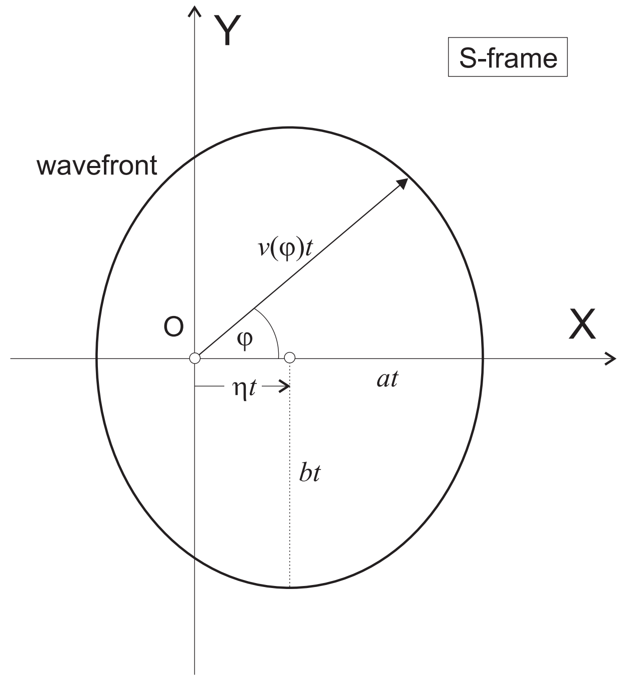

FIG. 1: An illustration of the light drag in the S-frame where the medium is moving at a constant speed u ($< c/n$) in the direction of the positive x-axis. The motion of the medium is a cause for the wavefront of the pulse originating from O to be a dragged ellipse possessing an axial symmetry with respect to the x-axis. Here, the photon emitted from O at an angle φ will travel at a speed $v(\varphi)$.

u in the positive direction of the x-axis. In this way, Eq. (1) is transformed into:

$$\frac{(x - \eta t)^2}{(at)^2} + \frac{y^2}{(bt)^2} = 1, \tag{3}$$

where by η, a and b we denoted:

$$\eta = u\left(\frac{1 - 1/n^2}{1 - u^2/n^2c^2}\right), \tag{4}$$

$$a = (c/n)\left(\frac{1 - u^2/c^2}{1 - u^2/n^2c^2}\right), \tag{5}$$

$$b = (c/n)\left(\frac{1 - u^2/c^2}{1 - u^2/n^2c^2}\right)^{1/2}. \tag{6}$$

According to Eq. (3), the wavefront of the pulse (the ray surface) with respect to an observer in S-frame is an expanding ellipse "dragged" by the moving medium (see Fig. 1). At time t from the emission of the pulse, the product ηt denotes the dragging parameter, that is, the distance from the origin O of the pulse (the origin of the xy-coordinate system) to the center of the ellipse, and the products at and bt are the semi-minor and semi-major axes of the ellipse, respectively. Consequently, the parameters a and b are the speeds of expansion of the semi-minor and semi-major axes of the ellipse, respectively, and η is the speed of the center of the ellipse along the x-axis. When $\eta < a$, the ellipse will be incompletely "dragged", and the origin of the pulse will remain inside the ellipse at all times. This is the case of a subluminal motion of the medium, when the speed of the medium u is smaller than c/n. On the other hand, when $\eta \geq a$, the "dragging" is overwhelming, and the ellipse no longer encloses the origin. In this case the medium is moving at "superluminal" speeds ($c/n < u < c$), causing an existence of a Mach cone, outside of which no light signal can ever reach, and inside of which light is doubled up, with two pulses of light in each direction [29, 32, 33].

We may translate Eq. (3) into the polar coordinates by making the substitutions $x = vt\cos\varphi$ and $y = vt\sin\varphi$:

$$\left(1 - \frac{u^2}{c^2}\sin^2\varphi - \frac{u^2}{n^2c^2}\cos^2\varphi\right)v^2$$
$$-2u\cos\varphi\left(1 - \frac{1}{n^2}\right)v + u^2 - \frac{c^2}{n^2} \;=\; 0. \qquad (7)$$

Here, v is the speed of the photon along a light ray directed at an angle φ from the velocity of the medium (see Fig. 1). Equation (7) is a quadratic in v, and it can be solved for v in terms of the angle φ, the speed of the medium u, and the refractive index n of the medium at rest. By rejecting the second solution of Eq. (7) from the requirement that $v = c/n$ when $u = 0$, we obtain:

$$v(\varphi) \;=\; \Big\{u(1 - 1/n^2)\cos\varphi + \left[c^2/n^2 - u^2(\sin^2\varphi\right.$$
$$+\;\left.\cos^2\varphi/n^2)\right]^{1/2}\left[1 - u^2/c^2\right]^{1/2}\Big\}$$
$$\times\;\left[1 - u^2/c^2(\sin^2\varphi + \cos^2\varphi/n^2)\right]^{-1}. \qquad (8)$$

From Eq. (8) we conclude that the ray velocity in the moving medium depends on the angle φ between the direction of the ray and the velocity of the medium. In this sense, an optical medium that is optically isotropic in its rest frame will possess an optical anisotropy in the frame in which it is moving at a constant velocity. This induced optical anisotropy is of a purely relativistic origin, and it is different from the usual anisotropy in the crystals [34].

The observer in the S-frame may track the path of a light ray in the moving medium by considering it to be a "stationary" medium that possesses a relativistically induced optical anisotropy characterized by an effective refractive index $n_{ef}(\varphi)$:

$$n_{ef}(\varphi) = \frac{c}{v(\varphi)}. \qquad (9)$$

Recalling Eq. (8) for $v(\varphi)$, the expression for n_{ef} in Eq. (9) can be re-written in terms of u, n and φ:

$$n_{ef}(\varphi) \;=\; \Big\{(u/c)(1 - n^2)\cos\varphi + n[1 - u^2/c^2]^{1/2}$$
$$\times\;\left[(u/c)^2(n^2 - 1)\cos^2\varphi + 1 - n^2u^2/c^2\right]^{1/2}\Big\}$$
$$\times\;\left(1 - n^2u^2/c^2\right)^{-1}. \qquad (10)$$

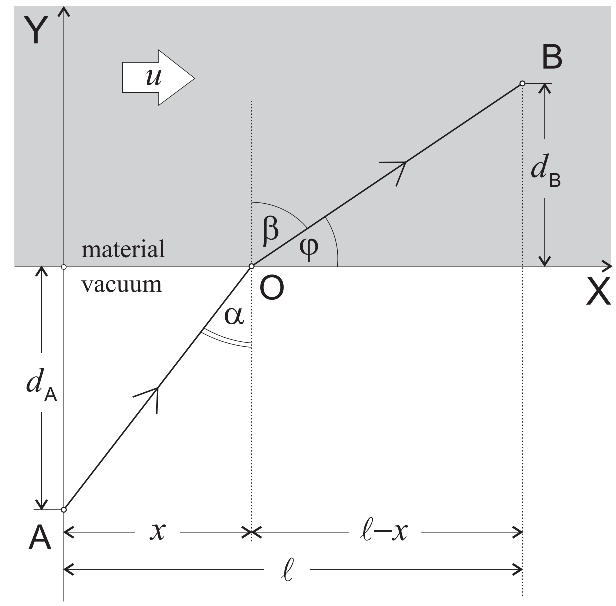

FIG. 2: The least-time derivation of the refraction formula when the material half-space is moving at a constant speed u in the direction of the positive x-axis.

When $n = 1$, Eqs. (8) and (10) reduce to $n_{ef} = 1$ and $v = c$ for all φ, which describes an expansion of the pulse in a vacuum. Also, if $u = 0$, then $n_{ef} = n$ and $v = c/n$ for all φ, which is the case when the medium is stationary.

If the photon moves parallel to the velocity of the medium, $n_{ef} \approx n \pm u/c(1-n^2)$ and $v \approx c/n \pm u(1-1/n^2)$ to the first order in u/c, where the sign is taken "+" or "–" depending on whether $\varphi = 0°$ or $\varphi = 180°$, respectively. This longitudinal light drag has been theoretically predicted by Fresnel (1818) on the basis of his ether theory, and experimentally confirmed by Fizeau (1851) in his celebrated running-water experiment [32].

III. REFRACTION FROM A MATERIAL HALF-SPACE MOVING PARALLEL TO THE INTERFACE

A photon traveling in vacuum is incident upon the surface of a semi-infinite optical material which occupies the positive region of the y-axis (see Fig. 2). The material half-space is moving at a constant speed u in the positive direction of the x-axis, parallel to the vacuum-material interface. The material in its rest frame has a refractive index n, and is assumed to be homogeneous, isotropic and non-dispersive. We will use Fermat's principle to derive the formula for the law of refraction of the photon from the interface between the vacuum and the moving medium.

Let A be the space-point in the vacuum half-space, belonging to the incident ray of the photon, and B is the space-point in the moving medium that lies on the refracted ray. By d_A and d_B we denote the shortest distances between the interface and the points A and B, respectively, and ℓ is the distance between the orthogonal projections of the points A and B on the interface. According to Fermat's principle, the true path between the two fixed points A and B taken by the photon is the one that is traversed in the least time. Of all the points along the interface, the photon will be refracted at the point O whose x-position minimizes the total time t_{AB} required for the photon to cover the path from A to B:

$$t_{AB} = t_{AO} + t_{OB}. \tag{11}$$

Here,

$$t_{AO} = \frac{\left(x^2 + d_A^2\right)^{1/2}}{c} \tag{12}$$

is the time for the photon to travel from A to O at a speed of light c, and

$$t_{OB} = \frac{\left[(\ell - x)^2 + d_B^2\right]^{1/2}}{v(\varphi)} \tag{13}$$

is the corresponding time from O to B. By $v(\varphi)$ we denote the speed of the photon along the refracted ray at an angle φ from the velocity of the medium. If we take into account that $v(\varphi) = c/n_{ef}(\varphi)$, and use Eqs. (12) and (13), we may re-write the total transit time t_{AB} in Eq. (11) as:

$$t_{AB} = \frac{\left(x^2 + d_A^2\right)^{1/2}}{c} + \frac{n_{ef}(\varphi)}{c}\left[(\ell - x)^2 + d_B^2\right]^{1/2}. \tag{14}$$

We substitute n_{ef} from Eq. (10) into Eq. (14), use the relation

$$\cos\varphi = \frac{\ell - x}{\left[(\ell - x)^2 + d_B^2\right]^{1/2}} \tag{15}$$

from Fig. 2, and simplify the result to obtain:

$$\begin{aligned} t_{AB} &= \frac{\left(x^2 + d_A^2\right)^{1/2}}{c} + \frac{(u/c^2)(1 - n^2)}{1 - n^2u^2/c^2}(\ell - x) \\ &+ \frac{(n/c)\left(1 - u^2/c^2\right)^{1/2}}{1 - n^2u^2/c^2}\Bigg\{(u/c)^2(n^2 - 1)(\ell - x)^2 \\ &+ (1 - n^2u^2/c^2)\left[(\ell - x)^2 + d_B^2\right]\Bigg\}^{1/2}. \end{aligned} \tag{16}$$

The principle of least time requires that $dt_{AB}/dx = 0$, and thus:

$$\begin{aligned} 0 &= \frac{x}{\left(x^2 + d_A^2\right)^{1/2}} - \frac{(u/c)(1 - n^2)}{1 - n^2u^2/c^2} - \frac{n(1 - u^2/c^2)^{3/2}}{1 - n^2u^2/c^2} \\ &\times \frac{(\ell - x)}{\left[(\ell - x)^2 + d_B^2\right]^{1/2}}\Bigg\{\frac{(u/c)^2(n^2 - 1)(\ell - x)^2}{\left[(\ell - x)^2 + d_B^2\right]} \\ &+ 1 - n^2u^2/c^2\Bigg\}^{-1/2}. \end{aligned} \tag{17}$$

From Fig. 2 we notice the expressions for the angle of incidence α and the angle of refraction β of the photon:

$$\sin\alpha = \frac{x}{\left(x^2 + d_A^2\right)^{1/2}}, \tag{18}$$

$$\sin\beta = \frac{\ell - x}{\left[(\ell - x)^2 + d_B^2\right]^{1/2}}, \tag{19}$$

which we use into Eq. (17) to get:

$$\begin{aligned} 0 &= \sin\alpha - \frac{(u/c)(1 - n^2)}{1 - n^2u^2/c^2} - \frac{n(1 - u^2/c^2)^{3/2}}{1 - n^2u^2/c^2} \\ &\times \frac{\sin\beta}{\left[(u/c)^2(n^2 - 1)\sin^2\beta + 1 - n^2u^2/c^2\right]^{1/2}}. \end{aligned} \tag{20}$$

If we employ the identity $\sin\beta = \tan\beta(1 + \tan^2\beta)^{-1/2}$ into Eq. (20) and do some algebra, we obtain a quadratic equation in $\tan\beta$:

$$\begin{aligned} &\left[n^2\left(1 - \frac{u}{c}\sin\alpha\right)^2 - \left(\sin\alpha - \frac{u}{c}\right)^2\right]\tan^2\beta - \frac{1}{1 - u^2/c^2} \\ &\times \left[\left(1 - \frac{n^2u^2}{c^2}\right)\sin\alpha + \frac{u}{c}(n^2 - 1)\right]^2 = 0. \end{aligned} \tag{21}$$

Equation (21) has two solutions in $\tan\beta$, and the solution:

$$\begin{aligned} \tan\beta &= \left[(n^2 - 1)(u/c) + (1 - n^2u^2/c^2)\sin\alpha\right] \\ &\times \left\{n^2\left[1 - (u/c)\sin\alpha\right]^2 - (\sin\alpha - u/c)^2\right\}^{-1/2} \\ &\times \left(1 - u^2/c^2\right)^{-1/2} \end{aligned} \tag{22}$$

is the only one that is physically correct, which we choose from the requirement that when $u = 0$ it should reduce to the usual Snell's law of refraction.

Equation (22) is the formula for the law of refraction of the photon from the vacuum-material interface when the optical material is moving uniformly parallel to its surface. It gives the angle of refraction β of the photon as a function of the incident angle α, the speed u of the medium, and the refractive index n of the medium in its rest frame. [Two alternative derivations leading to the same refraction formula are given in Ref. [32].]

The functional dependence of the angle of refraction β on the incident angle α is plotted in Fig. 3 for $n = 1.5$ and for several different values of the speed u of the medium. We immediately notice that the law of refraction in Eq. (22) differs considerably from the usual Snell's law in the case of a stationary medium. First of all, when $u/c > 0$, the angle of refraction β attain non-zero values even when the incident angle α is zero. In other words, the photon that enters the moving medium perpendicularly to its surface will be dragged by the medium in the moving direction. This is the transverse Fresnel-Fizeau light drag, theoretically predicted by Fresnel in 1818 and experimentally detected by Jones in the early 1970s with an accuracy up to the first order of u/c [32, 35, 36, 37].

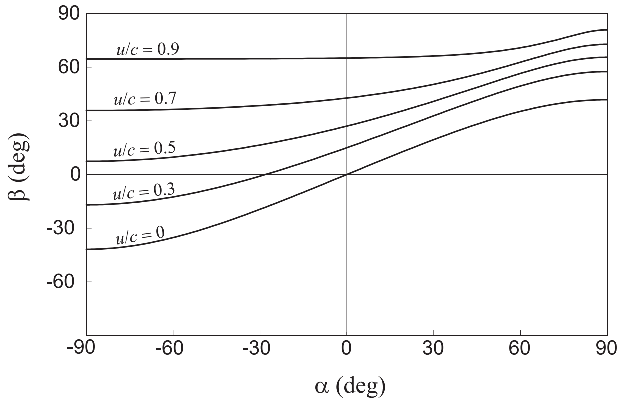

FIG. 3: The angle of refraction β versus the incident angle α for $n = 1.5$ and for different speeds u of the material half-space, when the material half-space is moving parallel to the interface.

The influence of the dragging effect on the refraction is also visible for non-zero incident angles, especially when they take negative values [38]. In the latter case, if $u/c > 0$, starting from $\alpha = 0°$ down to a certain negative incident angle, the angle of refraction β is positive (see Fig. 3), which means that both the incident and the refracted ray of the photon will lie on the same side with respect to the interface normal. Moreover, when the speed u of the medium becomes greater than c/n^2, the angle of refraction will be positive for any value of the incident angle α.

Equation (22) can also be used to calculate the angle of refraction of the photon when the medium moves in the opposite direction to the one in Fig. 2, that is, in the negative direction of the x-axis. However, the situation when the photon is incident at a positive angle α upon the medium which moves at a speed $-u$ is physically identical to the situation which corresponds to a negative incident angle $-\alpha$ and a speed u.

At the end, we emphasize that the absolute value of the angle of refraction β in Eq. (22) cannot become equal or greater than 90°, no matter what the values of α, n and u are. In other words, it is not possible to observe the situation in which the photon would slide along the surface of the medium, or to be reflected back to the vacuum half-space. To prove this property, we will try to find such a value of α which will turn the denominator in the right-hand side of Eq. (22) into zero. This is the condition for the angle β to equal $\pm 90°$. Hence, α should be such that:

$$n^2\left(1-\frac{u}{c}\sin\alpha\right)^2-\left(\sin\alpha-\frac{u}{c}\right)^2=0, \tag{23}$$

which is a quadratic equation in $\sin\alpha$, and has two solutions:

$$(\sin\alpha)_{1,2}=\frac{n\pm u/c}{nu/c\pm 1}. \tag{24}$$

Having in mind that $n > 1$ and $u < c$, we have $|(n \pm u/c)(nu/c \pm 1)^{-1}| > 1$, which means that there are no real values for the angle α to satisfy Eq. (23).

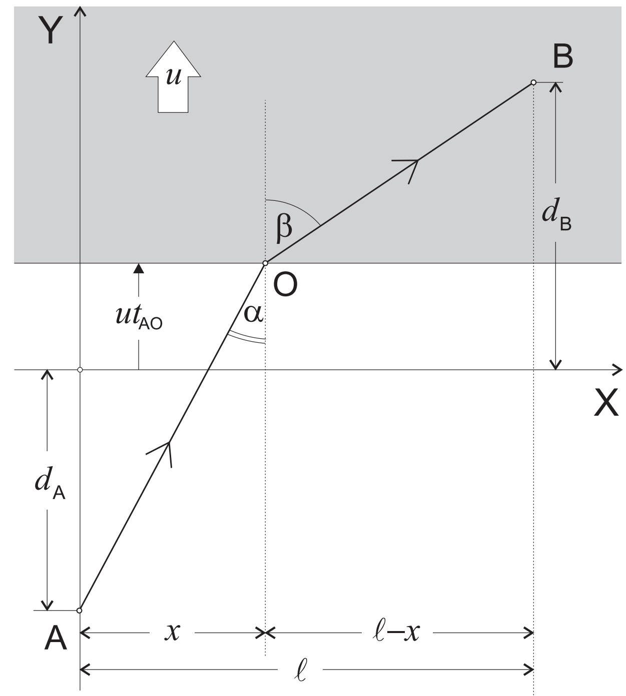

FIG. 4: The least-time derivation of the refraction formula when the material half-space is moving at a constant speed u in the positive direction of the y-axis.

IV. REFRACTION FROM A MATERIAL HALF-SPACE MOVING PERPENDICULAR TO THE INTERFACE

In the following, we consider the case in which the material half-space in Sec. III is moving at a constant speed u in the positive direction of the y-axis. The geometry of the problem is shown in Fig. 4. Observe that unlike the case in Sec. III, the vacuum-material interface also is in uniform motion, having the same constant speed u as the medium in the positive y-direction.

A photon is emitted from point A in the vacuum half-space. At this instant, the vacuum-material interface coincides with the x-axis, and d_A is the shortest distance between the interface and the point A. After being refracted from the moving interface, the photon reaches the fixed point B in the material half-space. We denote by d_B the shortest distance between the interface and the point B at time when the photon was emitted from A, and by ℓ the distance between the orthogonal projections of the points A and B on the interface.

Due to the motion of the vacuum-material interface, the hypothetical points from which the photon may be refracted at the interface will not lie on the same horizontal. According to Fermat's principle, the photon will be refracted from a point O that makes AOB the path of the shortest transit time (see Fig. 4). We denote by t_{AO} the time required for the photon to go from A to O, and by t_{OB} the time from O to B. Observe that when the photon reaches point O, the interface has already crossed the distance ut_{AO} in the positive y-direction from its position $y = 0$ when the photon was emitted from A.

The total time t_{AB} for the photon to travel the path AOB is:

$$t_{AB} = t_{AO} + t_{OB}. \tag{25}$$

From Fig. 4 we have:

$$(ct_{AO})^2 = (d_A + ut_{AO})^2 + x^2, \tag{26}$$

$$[v(\beta)t_{OB}]^2 = (d_B - ut_{AO})^2 + (\ell - x)^2, \tag{27}$$

where $v(\beta)$ is the speed of the photon in the moving medium in the direction determined by the angle β. The speed $v(\beta)$ of the photon follows from Eq. (8) if we simply substitute β for φ. In order to simplify the derivation of the refraction formula, we re-write Eqs. (26) and (27) as:

$$t_{AO} = \frac{1}{c}\left[(d_A + ut_{AO})^2 + x^2\right]^{1/2}, \tag{28}$$

$$t_{OB} = \frac{n_{ef}(\beta)}{c}\left[(d_B - ut_{AO})^2 + (\ell - x)^2\right]^{1/2}, \tag{29}$$

with $n_{ef}(\beta)$ being the effective refractive index of the moving half-space in this case:

$$\begin{aligned} n_{ef}(\beta) &= \Big\{(u/c)(1-n^2)\cos\beta + n[1-u^2/c^2]^{1/2} \\ &\times \left[(u/c)^2(n^2-1)\cos^2\beta + 1 - n^2u^2/c^2\right]^{1/2}\Big\} \\ &\times \left(1 - n^2u^2/c^2\right)^{-1}. \end{aligned} \tag{30}$$

Thus, the total transit time t_{AB} in Eq. (25) can be re-written as:

$$\begin{aligned} t_{AB} &= \frac{1}{c}\left[(d_A + ut_{AO})^2 + x^2\right]^{1/2} \\ &+ \frac{n_{ef}(\beta)}{c}\left[(d_B - ut_{AO})^2 + (\ell - x)^2\right]^{1/2}. \end{aligned} \tag{31}$$

If we substitute Eq. (30) into Eq. (31), and take into account the expression:

$$\cos\beta = \frac{d_B - ut_{AO}}{\left[(d_B - ut_{AO})^2 + (\ell - x)^2\right]^{1/2}}, \tag{32}$$

from Fig. 4, we obtain an expression for the transit time t_{AB} of the photon as a function of the position x of the point of refraction O at the moving interface:

$$\begin{aligned} t_{AB} &= \frac{\left[(d_A + ut_{AO})^2 + x^2\right]^{1/2}}{c} + \frac{u(1-n^2)(d_B - ut_{AO})}{c^2 - n^2u^2} \\ &+ \frac{(n/c)\left(1-u^2/c^2\right)^{1/2}}{1-n^2u^2/c^2}\Bigg\{\frac{u^2}{c^2}(n^2-1)(d_B - ut_{AO})^2 \\ &+ \left(1 - \frac{n^2u^2}{c^2}\right)\left[(d_B - ut_{AO})^2 + (\ell - x)^2\right]\Bigg\}^{1/2}. \end{aligned} \tag{33}$$

We now invoke the principle of least time that the value of x should be such that the transit time t_{AB} of the photon is minimum. We take the derivative of t_{AB} in Eq. (33) with respect to x and set the result to zero to obtain:

$$\begin{aligned} 0 &= u\cos\alpha\left(\frac{dt_{AO}}{dx}\right) + \sin\alpha + \frac{(u^2/c)(n^2-1)}{1-n^2u^2/c^2} \\ &\times \left(\frac{dt_{AO}}{dx}\right) - \frac{n\left(1-u^2/c^2\right)^{1/2}}{1-n^2u^2/c^2}\Bigg[u\left(1-\frac{u^2}{c^2}\right) \\ &\times \left(\frac{dt_{AO}}{dx}\right)\cos\beta + \left(1-\frac{n^2u^2}{c^2}\right)\sin\beta\Bigg] \\ &\times \left[\frac{u^2}{c^2}(n^2-1)\cos^2\beta + 1 - \frac{n^2u^2}{c^2}\right]^{-1/2}, \end{aligned} \tag{34}$$

where we employed the relations:

$$\sin\alpha = \frac{x}{\left[x^2 + (d_A + ut_{AO})^2\right]^{1/2}}, \tag{35}$$

$$\cos\alpha = \frac{d_A + ut_{AO}}{\left[x^2 + (d_A + ut_{AO})^2\right]^{1/2}}, \tag{36}$$

$$\sin\beta = \frac{\ell - x}{\left[(\ell - x)^2 + (d_B - ut_{AO})^2\right]^{1/2}}, \tag{37}$$

from Fig. 4, together with Eq. (32) and the fact that $t_{AO} = t_{AO}(x)$. To find dt_{AO}/dx, we take the derivative of Eq. (26) with respect to x and obtain:

$$2c^2 t_{AO}\left(\frac{dt_{AO}}{dx}\right) = 2(d_A + ut_{AO})u\left(\frac{dt_{AO}}{dx}\right) + 2x, \tag{38}$$

from which we get:

$$\left(\frac{dt_{AO}}{dx}\right) = \frac{x/(ct_{AO})}{c\left[1 - (u/c)(d_A + ut_{AO})/(ct_{AO})\right]}. \tag{39}$$

By recognizing from Fig. 4 that $\sin\alpha = x/(ct_{AO})$ and $\cos\alpha = (d_A + ut_{AO})/(ct_{AO})$, we recast Eq. (39) into the form:

$$\left(\frac{dt_{AO}}{dx}\right) = \frac{\sin\alpha}{c - u\cos\alpha}. \tag{40}$$

We substitute Eq. (40) into Eq. (34), apply the trigonometric identities $\sin\beta = \tan\beta(1+\tan^2\beta)^{-1/2}$ and $\cos\beta = (1+\tan^2\beta)^{-1/2}$, and do some algebra to obtain a quadratic equation in $\tan\beta$:

$$\begin{aligned} &\left\{\frac{n^2(1-n^2u^2/c^2)}{1-u^2/c^2} - \frac{\sin^2\alpha}{\left[1-(u/c)\cos\alpha\right]^2}\right\}\tan^2\beta \\ &+ \frac{2n^2(u/c)\sin\alpha}{1-(u/c)\cos\alpha}\tan\beta - \frac{(1-u^2/c^2)\sin^2\alpha}{\left[1-(u/c)\cos\alpha\right]^2} = 0. \end{aligned} \tag{41}$$

Equation (41) has two solutions in $\tan\beta$, and its only physically correct solution is

$$\tan\beta = (1-u^2/c^2)\sin\alpha \times \Big\{n^2(u/c)\left[1-(u/c)\cos\alpha\right]$$

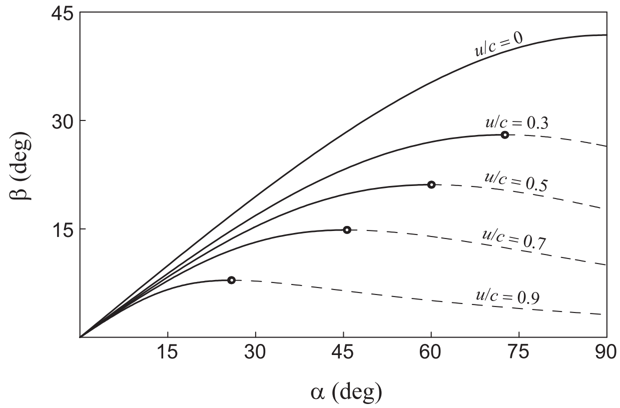

FIG. 5: The angle of refraction β versus the incident angle α for $n = 1.5$ and for different speeds u of the material half-space moving in the positive direction of the y-axis. The portions of the curves corresponding to $\alpha \geq \arccos(u/c)$ are drawn with dashed lines. At incident angles corresponding to the dashed curves, the refraction does not occur.

$$+\left[n^2\left[1-(u/c)\cos\alpha\right]^2-(1-u^2/c^2)\sin^2\alpha\right]^{1/2}\Bigg\}^{-1}, \tag{42}$$

which is the formula for the law of refraction of the photon in this case. We reject the second solution of Eq. (41) as a physically incorrect one, by using the same argument as in the deduction of Eq. (22) from Eq. (21). [A derivation of Eq. (42) via the relativistic velocity transformation formulas is provided in Appendix A.]

The obtained refraction formula works both for positive and negative values of the incident angle α, as well as for both moving directions of the medium. The right-hand side of Eq. (42) is an odd function of α, and hence the change in the sign of α does not introduce physically new situation. The path of the photon which is incident upon the surface at a negative angle $-\alpha$ will be symmetric with respect to the surface normal, to the path of the photon corresponding to a positive incident angle α. Therefore, in the following, we consider only positive incident angles ($0^\circ \leq \alpha < 90^\circ$).

When the medium moves in the positive direction of the y-axis, $u/c > 0$ and the right-hand side of Eq. (42) is always positive, which means that the angle of refraction β is between 0° and 90°. The dependence of β on the incident angle α for $n = 1.5$ and for various positive values of u/c is shown in Fig. 5. We see that as u/c varies from 0 to 1, the curves corresponding to different values of u/c are regularly shifted toward the smaller values of β, which is expectable if one takes into account the dragging effect of the moving medium. We emphasize that the portions of the curves drawn with dashed lines correspond to incident angles at which the photon will never reach the interface. Namely, for a given speed u of the medium in the positive y-direction, there exists an interval of values for the incident angle α ($\alpha_{\max} \leq \alpha < 90^\circ$) for which the y-projection of the photon's velocity ($c\cos\alpha$) is less than

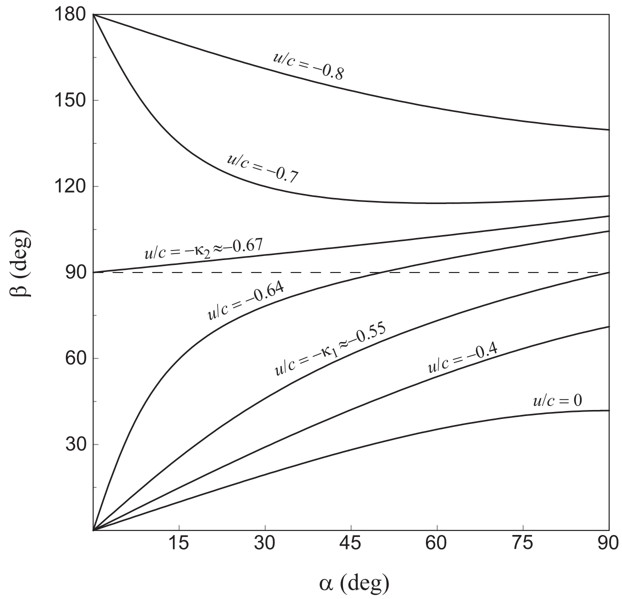

FIG. 6: The angle of refraction β as a function on the incident angle α for $n = 1.5$ when the material half-space is moving perpendicular to the interface, in the negative direction of the y-axis. When $-\kappa_1 < u/c < 0$ the refraction is regular, and the angle of refraction β of the photon is in the interval $[0^\circ, 90^\circ]$. For $-\kappa_2 < u/c < -\kappa_1$, the refraction is: a) regular ($0 \leq \beta < 90^\circ$), if $0 \leq \alpha < \alpha_c$; and b) backward ($90 < \beta < 180$), if $\alpha_c < \alpha < 90^\circ$. If $u/c < -\kappa_2$, the refraction is always backward.

or equal to the velocity of the medium. The value of $\alpha_{\max}$ follows from:

$$\cos\alpha_{\max} = u/c, \tag{43}$$

and it can be shown that it is exactly the incident angle at which the curve $\beta = \beta(\alpha)$ attains the maximum value. Hence, for incident angles belonging to the interval $[\alpha_{\max}, 90^\circ]$, the refraction becomes impossible. Translating to the frame in which the medium is at rest, and thus taking into account the aberration phenomenon, we have a photon "incident" upon the interface at angles larger than 90°.

The situation changes dramatically when $u/c < 0$. In this case, the denominator on the right-hand side of Eq. (42) can take zero or even negative values, which implies that the angle of refraction β can become equal or greater than 90°. In Fig. 6 we plotted β as a function of α for $n = 1.5$ and for various negative values of u/c. What we see in Fig. 6 can be summarized as follows. When the medium moves in the negative direction of the y-axis up to some value of $|u/c|$ denoted as κ_1, the refraction seems to be regular. In this range of speeds ($-\kappa_1 < u/c < 0$), as the absolute value of the speed increases, the curves are shifted toward higher values of β, and β remains smaller than 90° for each α. When $u/c < -\kappa_1$, a specific value for the incident angle α appears for which the angle of refraction of the photon becomes equal to 90°. We denote this critical incident angle by α_c. For $\alpha_c < \alpha < 90^\circ$, the angle of refraction β is greater than $90°$. By further increasing the absolute value of u/c, the critical angle α_c decreases, and for some value of $|u/c|$, denoted by κ_2, α_c becomes $0°$. For $u/c < -\kappa_2$, the angle of refraction β is greater than $90°$ for all values of α. The quantities κ_1 and κ_2 are functions on the refractive index n of the medium in its rest frame, and as n increases, the values of κ_1 and κ_2 are becoming closer to each other. It can be shown (see Appendix B) that $\kappa_1 = (1+n^2)^{-1/2}$ and $\kappa_2 = 1/n$ (in our case, for $n = 1.5$, $\kappa_1 \approx 0.55$ and $\kappa_2 \approx 0.67$). Evidently, the value κ_2 corresponds to the smallest superluminal speed of the medium.

The existence of angles of refraction greater than $90°$ in the case $u/c < 0$ implies that the photon, instead of moving in the same general direction as before the refraction, moves backward. At first sight, it might seem that the photon was reflected at the interface instead of being refracted, but this is not what had happened (see Fig. 7). Once the photon strikes the vacuum-material interface, it undergoes a refraction, enters the medium and continues to move into the medium at all times. The curious behavior of the refracted photon is due to the fact that the photon, while penetrating into the medium, is dragged by the overall motion of the medium. This dragging effect is a cause for the y-component of the photon's velocity in the moving medium to be negative, and hence, the refracted photon will move in the same general direction as the medium.

We emphasize that the path of the photon that undergoes a backward refraction does not coincide with the path which it would trace if it were reflected from the interface (see Fig. 7). The path of the backwardly refracted photon is fully determined by Eq. (42), and the path of the reflected photon obeys the Einstein's formula for reflection from a uniformly moving mirror (see Ref. [28, 39, 40]). In the language of modern quantum electrodynamics, whether the photon will be reflected or refracted at the inerface is a matter of probability. In this sense, if a beam of photons is incident on the moving interface at an angle α, and if the conditions for the appearance of the backward refraction are fulfilled, some of the photons will be reflected, and some backwardly refracted. Therefore, in a more realistic view in which the reflection is considered also, two different beams spreading backward will be identified. In this way, we have a full consistency between the observations with respect to both inertial frames, S and S'. Namely, for the observer in S'-frame to which the medium is at rest, both processes are developing in an ordinary way: the refracted photon moves forward, obeying the Snell's law of refraction, while the reflected one moves backward, according to the usual law of reflection: incident angle equals the reflected angle. Hence, the observer in S' clearly distinguishes two rays, and consequently, two rays must exist with respect to the observer in the S-frame: one reflected, and one backwardly refracted.

V. CONCLUDING REMARKS

We have presented a novel approach to the problem of refraction of a light ray at an interface between two homogeneous, isotropic and non-dispersive transparent optical materials in uniform rectilinear motion. Our method amalgamates the original Fermat's principle and the fact that an isotropic optical material at rest becomes optically anisotropic if it is moving at a constant velocity. The derivation is in the framework of basic optics and special relativity, requiring some knowledge of calculus at an elementary level. As such, it may be regarded as an instructive addendum to the typical introductory physics courses.

We have analyzed in details the refraction at a vacuum-material interface, when the material half-space is moving uniformly parallel or uniformly perpendicular to the interface. In both cases, we considered the plane of incidence to be normal to the interface and parallel to the velocity of the medium. The coincidence of the obtained refraction formulas with the ones obtained with the other methods confirms the validity of Fermat's principle in the presence of uniformly moving boundaries and media.

In addition, when the material half-space moves perpendicular to the interface, in the direction which is opposite to that of the incident light, we observed a "backward refraction" of light. To our knowledge, this optical effect has not been noticed before in the literature.

We emphasize that the refraction formulas obtained in this paper refer to refraction of a photon, and thus, describe the refraction of a light ray. We note that in a set of papers on the subject of reflection and transmission of a plane electromagnetic wave by a moving medium, starting with the pioneering papers by Tai and Yeh, the refraction formulas for different setups were derived by using the notions from the classical electromagnetic theory and the Lorentz transformation [41, 42, 43, 44, 45, 46, 47]. Although these publications also include the cases discussed in our Sections III and IV, the resulting refraction formulas do not match the refraction formulas in Eq. (22) and (42) in our paper [48]. This mismatch is due to the fact that Tai, Yeh and their successors considered the angles of incidence, reflection and refraction of the light wave as the angles between the corresponding propagation vectors and the interface normal. However, we have shown that an optical isotropic medium in uniform rectilinear motion becomes an optical anisotropic medium. Therefore, the propagation vector in the moving medium will, in general, not coincide with the light ray in the moving medium. In conclusion, the refraction formulas by Tai, Yeh and their followers do not refer to refraction of a light ray.

The approach in this paper can be extended to include more sophisticated situations, specifically, by taking into account the dispersion of the material, or considering various three-dimensional cases when the plane of incidence is not necessary normal to the interface and not parallel to the velocity of the medium. We also suggest a rep-

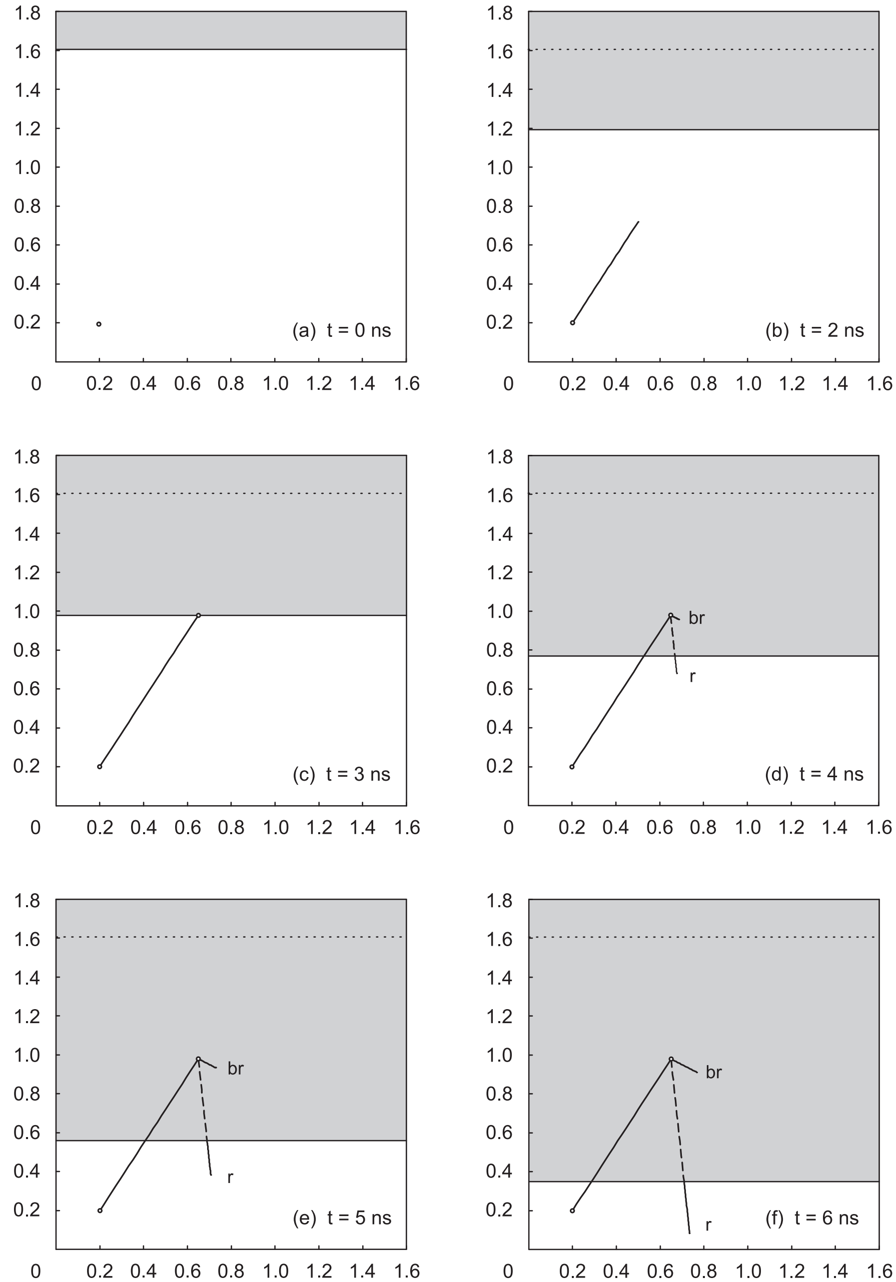

FIG. 7: A space-time simulation depicting the difference between the reflection and the backward refraction. The material half-space (the shaded region) is moving in the negative y-direction at a speed $0.7c$. The refractive index of the material half-space in its rest frame is $n = 1.5$. The values of x and y coordinates are given in meters. (a) The situation at time $t = 0$ ns. A photon, located at point $(0.2, 0.2)$, is incident upon the vacuum-material interface, making an angle of 30° with respect to the interface normal in the vacuum half-space. The interface is located at $y = 1.6084478$; (b) The path of the photon and the position of the interface at time $t = 2$ ns; (c) The situation at $t = 3$ ns. The photon reaches the interface at point $(0.649689, 0.97884)$; (d) The situation at $t = 4$ ns. The ray br corresponds to the path of the backwardly refracted photon, making an angle of 119.974° with respect to the interface normal in the material half-space. The speed of the backwardly refracted photon is $0.151c$, and its y velocity component is $-0.07544c$. The ray r is the path of the photon if it were reflected from the interface. The angle of reflection is -5.41445° with respect to the interface normal in the vacuum half-space. The y velocity component of the reflected photon is $-0.99554c$; (e) The situation at $t = 5$ ns; (f) The situation at $t = 6$ ns.

etition of the derivations in Sec. III and IV when the refraction occurs at an interface between two uniformly moving non-vacuum optical materials having different refractive indices in their rest frames of reference.

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APPENDIX A: A DERIVATION OF EQ. (42) BY USING THE RELATIVISTIC VELOCITY TRANSFORMATION FORMULAS

The refraction formula in Eq. (42) can also be obtained by comparing the path of the photon in the S'- frame where the medium is stationary, to the corresponding path of the photon in the S-frame where the medium is in uniform rectilinear motion. We first describe the reflection with respect to S'-frame. Let the vacuum-material interface coincide with x'-axis, with $y' > 0$ being the region of the material half-space, and $y' < 0$ the region of the vacuum half-space. Let the photon be incident on the interface from the vacuum half-space in a direction making a positive angle α' with the interface normal. The photon is refracted in the material half-space at an angle β' with respect to the interface normal, obeying the Snell's law of refraction:

$$\sin\alpha' = n\sin\beta', \tag{A1}$$

where n is the refractive index of the medium at rest. Taking that the incident photon moves at a speed of light in vacuum c, and the refracted photon at a speed c/n, the velocity components of the photon in the S'-frame are:

$$v'_{x_1} = c\sin\alpha', \tag{A2}$$
$$v'_{y_1} = c\cos\alpha', \tag{A3}$$
$$v'_{x_2} = (c/n)\sin\beta', \tag{A4}$$
$$v'_{y_2} = (c/n)\cos\beta', \tag{A5}$$

where index "1" corresponds to the components of the incident photon, and index "2" to the components of the refracted photon (for simplicity, we use v'_x and v'_y instead of $v'_{x'}$ and $v'_{y'}$). With respect to S-frame where the material half-space is moving at a constant speed u in the positive y-direction, the photon is incident on the moving interface at an angle α at the same constant speed c as in the S'-frame, but its speed after being refracted from the interface is v_β which is a function of the angle of refraction β. Thus, the corresponding velocity components of the photon in S-frame are:

$$v_{x_1} = c\sin\alpha, \tag{A6}$$
$$v_{y_1} = c\cos\alpha, \tag{A7}$$
$$v_{x_2} = v_\beta\sin\beta, \tag{A8}$$
$$v_{y_2} = v_\beta\cos\beta. \tag{A9}$$

Dividing Eq. (A8) with Eq. (A9), and using the relativistic transformation formulas for the velocity components between S and S':

$$v_x = \frac{v'_x(1-u^2/c^2)^{1/2}}{1+v'_y u/c^2}, \tag{A10}$$
$$v_y = \frac{v'_y+u}{1+v'_y u/c^2}, \tag{A11}$$

we obtain:

$$\tan\beta = \frac{v'_{x_2}(1-u^2/c^2)^{1/2}}{v'_{y_2}+u}. \tag{A12}$$

Combining Eqs. (A4) and (A5) with Eq. (A1), we obtain:

$$v'_{x_2} = (c/n^2)\sin\alpha', \tag{A13}$$
$$v'_{y_2} = (c/n)(1-\sin^2\alpha'/n^2)^{1/2}, \tag{A14}$$

which we substitute into Eq. (A12) to get:

$$\tan\beta = \frac{(c/n^2)\sin\alpha'(1-u^2/c^2)^{1/2}}{(c/n)(1-\sin^2\alpha'/n^2)^{1/2}+u}. \tag{A15}$$

Using the velocity transformation formula:

$$v'_x = \frac{v_x(1-u^2/c^2)^{1/2}}{1-v_y u/c^2} \tag{A16}$$

and Eqs. (A2), (A6) and (A7), we get:

$$\sin\alpha' = \frac{\sin\alpha(1-u^2/c^2)^{1/2}}{1-(u/c)\cos\alpha}. \tag{A17}$$

We substitute Eq. (A17) into Eq. (A15) and simplify the result to obtain the law of refraction of the photon in Eq. (42).

APPENDIX B: CONDITIONS FOR THE APPEARANCE OF THE BACKWARD REFRACTION

Because a backward refraction occurs only for $u/c < 0$, we rewrite Eq. (42) by putting $-\kappa$ instead of u/c ($\kappa = |u/c|$):

$$\tan\beta = (1-\kappa^2)\sin\alpha \times \Big\{-\kappa n^2\,(1+\kappa\cos\alpha) + \Big[n^2\,[1+\kappa\cos\alpha]^2 - (1-\kappa^2)\sin^2\alpha\Big]^{1/2}\Big\}^{-1}. \tag{B1}$$

The photon will be refracted at $\beta = 90^\circ$ when $\alpha = \alpha_c$, which is the case when the denominator in the right-hand side of Eq. (B1) equals zero:

$$\left[n^2\left[1+\kappa\cos\alpha_c\right]^2 - (1-\kappa^2)(1-\cos^2\alpha_c)\right]^{1/2} = \\ = \kappa n^2\left(1+\kappa\cos\alpha_c\right). \quad \text{(B2)}$$

The last equation is an irrational equation in $\cos\alpha_c$, and its solution is equivalent to the solution of the system:

$$\left\{\begin{array}{rcl} \left[1-\kappa^2+\kappa^2 n^2\left(1-\kappa^2 n^2\right)\right]\cos^2\alpha_c & & \\ +2\kappa n^2\left(1-\kappa^2 n^2\right)\cos\alpha_c & & \\ +1-\kappa^2-n^2\left(1-\kappa^2 n^2\right) & = & 0 \\ \kappa n^2\left(1+\kappa\cos\alpha_c\right) & \geq & 0 \end{array}\right. \quad \text{(B3)}$$

which simplifies to:

$$\left\{\begin{array}{rcl} (\cos\alpha_c)_{1,2} & = & \Big\{-\kappa n^2(1-\kappa^2 n^2) \pm (1-\kappa^2) \\ & & \times\left[1-n^2(1-\kappa^2 n^2)\right]^{1/2}\Big\} \\ & & \times\left[1-\kappa^2+\kappa^2 n^2(1-\kappa^2 n^2)\right]^{-1} \\ \cos\alpha_c & \geq & -1/\kappa \end{array}\right. \quad \text{(B4)}$$

The negative values for $\cos\alpha_c$ imply $\alpha_c > 90^\circ$. Since these solutions are physically unacceptable, we reject the solution with the "–" sign in the first equation of the system (B4), because this solution corresponds to $\cos\alpha_c < 0$. What is left to be further analyzed is the expression:

$$\begin{aligned} \cos\alpha_c \; = \; & \Big\{-\kappa n^2(1-\kappa^2 n^2) + (1-\kappa^2) \\ & \times\left[1-n^2(1-\kappa^2 n^2)\right]^{1/2}\Big\} \\ & \times\left[1-\kappa^2+\kappa^2 n^2(1-\kappa^2 n^2)\right]^{-1}. \end{aligned} \quad \text{(B5)}$$

The right-hand side in Eq. (B5) is a real number if the expression under the square-root is non-negative, i.e. $1 - n^2(1-\kappa^2 n^2) \geq 0$. Hence $\kappa \geq (n^2-1)^{1/2}/n^2 = \kappa_{\min}$. Specifically, when $\kappa = \kappa_{\min}$, Eq. (B5) reduces to $\cos\alpha_c = -(n^2-1)^{1/2}/n^2$. Since $0^\circ \leq \alpha_c \leq 90^\circ$, the right-hand side in Eq. (B5) should satisfy the inequality:

$$\begin{aligned} 0 \; \leq \; & \Big\{-\kappa n^2(1-\kappa^2 n^2) + (1-\kappa^2) \\ & \times\left[1-n^2(1-\kappa^2 n^2)\right]^{1/2}\Big\} \\ & \times\left[1-\kappa^2+\kappa^2 n^2(1-\kappa^2 n^2)\right]^{-1} \leq 1. \end{aligned} \quad \text{(B6)}$$

For a fixed n, one can determine the interval of values for κ, $\kappa \in [\kappa_1, \kappa_2]$, such that for each κ within this interval, Eq. (B5) has a physically acceptable solution for α_c. This is the critical incident angle at which the backward refraction actually appears, and for $\alpha < \alpha_c$ the refraction is regular (forward), while for $\alpha > \alpha_c$ the refraction is backward. The ends of the interval $[\kappa_1, \kappa_2]$ correspond to the critical incident angles $\alpha_c = 90^\circ$ and $\alpha_c = 0^\circ$, respectively, and can be obtained from the following equations:

$$\left(1-\kappa_1^2\right)\left[1-n^2(1-\kappa_1^2 n^2)\right]^{1/2} = \kappa_1^2 n^2\left(1-\kappa_1^2 n^2\right), \quad \text{(B7)}$$

$$\left[1-n^2(1-\kappa_2^2 n^2)\right]^{1/2} = \frac{1-\kappa_2\left[1-n^2\left(1-\kappa_2^2 n^2\right)\right]}{1-\kappa_2}. \quad \text{(B8)}$$

The last two equations are irrational equations in $\kappa_{1,2}$. To avoid cumbersome calculations, we substitute $x' = n^2(1-\kappa_1^2 n^2)$ in Eq. (B7) and $x'' = 1-n^2(1-\kappa_2^2 n^2)$ in Eq. (B8) to obtain:

$$(1-x')^{1/2} = \frac{\kappa_1 x'}{1-\kappa_1^2}, \quad \text{(B9)}$$

$$x''^{1/2} = \frac{1-\kappa_2 x''}{1-\kappa_2}. \quad \text{(B10)}$$

Equation (B9) is equivalent to the following system:

$$\left\{\begin{array}{rcl} \kappa_1^2 x'^2 + \left(1-\kappa_1^2\right)x' - 1 + \kappa_1^2 & = & 0 \\ x' & \geq & 0 \end{array}\right. \quad \text{(B11)}$$

while Eq. (B10) corresponds to:

$$\left\{\begin{array}{rcl} \kappa_2^2 x''^2 - \left(1+\kappa_2^2\right)x'' + 1 & = & 0 \\ x'' & \leq & 1/\kappa_2 \end{array}\right. \quad \text{(B12)}$$

The quadratic equation in Eq. (B11) has two solutions in x': $x'_1 = 1-\kappa_1^2$ and $x'_2 = -(1-\kappa_1^2)/2\kappa_1^2$. The solution x'_2 is not consistent with the inequality in Eq. (B11), since $x'_2 < 0$ (we have taken into account that $0 < \kappa < 1$). Thus, the only solution of the system (B11) is $x'_1 = 1-\kappa_1^2$, leading to the equation $n^2(1-\kappa_1^2 n^2) = 1-\kappa_1^2$, which gives:

$$\kappa_1 = (1+n^2)^{-1/2}. \quad \text{(B13)}$$

Similarly, by solving the quadratic equation in Eq. (B12) one obtains two solutions in x'': $x''_1 = 1$ and $x''_2 = 1/\kappa_2^2$. Since the solution x''_2 does not satisfy the inequality in Eq. (B12), we reject it and consider only the solution $x''_1 = 1$, which leads to $1-n^2(1-\kappa_2^2 n^2) = 1$, and hence:

$$\kappa_2 = 1/n. \quad \text{(B14)}$$

Equations (B13) and (B14) determine the length of the interval $[\kappa_1, \kappa_2]$ and its dependence on n. We can now summarize the results. For $\kappa < (1+n^2)^{-1/2}$, the refraction is regular (forward) for any incident angle α. When $(1+n^2)^{-1/2} \leq \kappa \leq 1/n$, there exists a critical incident angle α_c, such that for α in the interval $0^\circ < \alpha < \alpha_c$ the refraction is regular (forward), while for $\alpha_c < \alpha < 90^\circ$ the refraction is backward. The angle α_c can be calculated from Eq. (B5). It depends on κ, and tends to zero as κ approaches $1/n$. When $\kappa = 1/n$, $\alpha_c = 0^\circ$, and the right-hand side of Eq. (B1) becomes an undetermined expression of type 0/0, which can be resolved by using L'Hospital's rule. When $\kappa > 1/n$, the denominator in Eq. (B1) is negative for any α. On the other hand, the numerator in Eq. (B1) is always non-negative. Therefore, for $\kappa > 1/n$, $\beta > 90°$ for each α. Hence, when the medium is moving in the negative direction of the y-axis, there exists a critical speed for the medium $|u_c| = \kappa_2 c = c/n$, corresponding to the treshold for the "superluminal" speeds. Beyond this critical speed, the photon will always be backwardly refracted.

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formulas in Eq. (22) and (42) above are describing the refraction of a light ray. The light ray corresponds to the group velocity, and correctly describes the path followed by light if the experiment is performed with a pencil of laser light. See, e.g., Ref. [34] and [37].